Lecture 6: Exercise 4

Explanation

Here we are practicing deriving the Lagrangian.

Hint

The process is to derive the Lagrangian for given conditions:

1. Determine the coordinates.
2. Calculate the generalized velocities.
3. Calculate the kinetic energy.
4. Calculate the potential energy.
5. Calculate the Lagrangian.
6. Determine the equations of motion by applying the Euler-Lagrange equation.

Answer

We begin be writing coordinates

$$X = R\cos\theta,\ \ Y = R\sin\theta$$

assuming $\theta = \theta(t)$, the velocities are now

$$\dot{X} = \dot{R}\cos\theta - R\,\dot{\theta}\sin\theta,\ \ \dot{Y} = \dot{R}\ \sin\theta + R\,\dot{\theta}\cos\theta$$

Since $T = \frac{1}{2}m\,v^2$, where we can use Pythagoras' theorem

$$v^2 = \dot{X}^2 + \dot{Y}^2 = \left(\dot{R}\cos\theta - R\,\dot{\theta}\sin\theta\right)^2 + \left(\dot{R}\ \sin\theta + R\,\dot{\theta}\cos\theta\right)^2$$

$$= \dot{\theta}^2\ R^2\sin^2(\theta) - 2\,\dot{\theta}\,R\,\dot{R}\sin(\theta)\cos(\theta) + \dot{R}^2\ \cos^2(\theta) + \dot{\theta}^2\ R^2\cos^2(\theta) - 2\,\dot{\theta}\,R\,\dot{R}\sin(\theta)\cos(\theta) + \dot{R}^2\ \sin^2(\theta)$$

$$= \dot{\theta}^2\ \left[R^2\sin^2(\theta) + R^2\cos^2(\theta)\right] - 2\,\dot{\theta}\left[R\,\dot{R}\sin(\theta)\cos(\theta) - R\,\dot{R}\sin(\theta)\cos(\theta)\right] + \dot{R}^2\ \left[\cos^2(\theta) + \sin^2(\theta)\right]$$

$$= \dot{\theta}^2\ \left\{R^2\left[\sin^2(\theta) + \cos^2(\theta)\right]\right\} + \dot{R}^2$$

$$= \dot{\theta}^2\ R^2\ + \dot{R}^2$$

The kinetic energy is then

$$T = \frac{1}{2}m\,v^2 = \frac{1}{2}m\left(\dot{\theta}^2\ R^2\ + \dot{R}^2\right)$$

Since we have a potential in polar coordinates, it is only dependent on the distance from the center, $V = V(R)$, so this gives us the Lagrangian

$$L = \frac{1}{2}m\left(\dot{\theta}^2\ R^2\ + \dot{R}^2\right) + V(R)$$

Since we are in polar coordinates $(R,\ \theta)$, we take the partial derivatives

$$\frac{\partial L}{\partial R} = m\,\dot{\theta}^2\ R\ + \frac{\partial V}{\partial R} = m\,\dot{\theta}^2\ R\ -\ F$$

and

$$\frac{\partial L}{\partial \theta} = 0$$

We also take the partial derivatives

$$\frac{\partial L}{\partial \dot{R}} = m\,\dot{R}$$

and

$$\frac{\partial L}{\partial \dot{\theta}} = m\,\dot{\theta}\,R^2$$

so

$$\frac{d}{d\,t}\left(\frac{\partial L}{\partial \dot{R}}\right) = m\,\ddot{R}$$

and

$$\frac{d}{d\,t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m\,R^2\,\ddot{\theta}$$

We can then write Lagrange's equations

$$\frac{d}{d\,t}\left(\frac{\partial L}{\partial \dot{X}_i}\right) = \frac{\partial L}{\partial X_i}$$

as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{R}}\right) = \frac{\partial L}{\partial R} \Rightarrow m\,\ddot{R} = m\,\dot{\theta}^2\,R - F \Rightarrow F = m\left(\ddot{R} - \dot{\theta}^2\,R\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} \Rightarrow m\,R^2\,\ddot{\theta} = 0$$