Lecture 6: Exercise 3

Explanation

Here we are practicing deriving the equation of motion from a given Lagrangian.

Hint

The process is to turn the crank of the machine that is the Lagrangian Method given the Lagrangian:

1. Calculate $\frac{\partial L}{\partial x_i}$
2. Calculate $\frac{\partial L}{\partial \dot{x}_i}$
3. Calculate $\frac{d}{d\,t}\left(\frac{\partial L}{\partial \dot{x}_i}\right)$
4. Substitute the calculated values into the Euler-Lagrange Equation: $\frac{d}{d\,t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i}$

Answer

We begin be writing the Lagrangian, Eq. (12)

$$L = \frac{m}{2}\left(\dot{X}^2 + \dot{Y}^2\right) + \frac{m\,\omega^2}{2}\left(X^2 + Y^2\right) + m\,\omega\left(\dot{X}\,Y - \dot{Y}\,X\right)$$

We now need to calculate $\frac{\partial L}{\partial x_i}$, since we have X and Y we write

$$\frac{\partial L}{\partial X} = \frac{\partial}{\partial X}\left[\frac{m}{2}\left(\dot{X}^2 + \dot{Y}^2\right) + \frac{m\,\omega^2}{2}\left(X^2 + Y^2\right) + m\,\omega\left(\dot{X}\,Y - \dot{Y}\,X\right)\right]$$

this becomes

$$\frac{\partial L}{\partial X} = \frac{\partial}{\partial X}\left[\frac{m\,\omega^2}{2}\left(X^2 + Y^2\right) + m\,\omega\left(\dot{X}\,Y - \dot{Y}\,X\right)\right]$$

We use the sum rule

$$\frac{\partial L}{\partial X} = \frac{\partial}{\partial X}\,\frac{m\,\omega^2}{2}\left(X^2 + Y^2\right) + \frac{\partial}{\partial X}\,m\,\omega\left(\dot{X}\,Y - \dot{Y}\,X\right)$$

We can treat $m\,\omega$ and $m\,\omega^2/2$ as constants and use the constant multiple rule

$$\frac{\partial L}{\partial X} = \frac{m\,\omega^2}{2}\,\frac{\partial}{\partial X}\left(X^2 + Y^2\right) + m\,\omega\,\frac{\partial}{\partial X}\left(\dot{X}\,Y - \dot{Y}\,X\right)$$

we can expand this using the sum rule,

$$\frac{\partial L}{\partial X} = \frac{m\,\omega^2}{2}\,\frac{\partial}{\partial X}\,X^2 + \frac{m\,\omega^2}{2}\,\frac{\partial}{\partial X}\,Y^2 + m\,\omega\,\frac{\partial}{\partial X}\,\dot{X}\,Y - m\,\omega\,\frac{\partial}{\partial X}\,\dot{Y}\,X$$

if there is no X the third term vanishes,

$$\frac{\partial L}{\partial X} = \frac{m\,\omega^2}{2}\,\frac{\partial}{\partial X}\,X^2 - m\,\omega\,\frac{\partial}{\partial X}\,\dot{Y}\,X$$

we again apply the constant multiple rule,

$$\frac{\partial L}{\partial X} = \frac{m\,\omega^2}{2}\,\frac{\partial}{\partial X}\,X^2 - m\,\omega\,\dot{Y}\,\frac{\partial}{\partial X}\,X$$

since $\partial X/\partial X = 1$, and we apply the power rule,

$$\frac{\partial L}{\partial X} = m\,\omega^2\,X - m\,\omega\,\dot{Y}$$

We do the same thing for Y,

$$\frac{\partial L}{\partial Y} = \frac{\partial}{\partial Y}\left[\frac{m}{2}\left(\dot{X}^2 + \dot{Y}^2\right) + \frac{m\,\omega^2}{2}\left(X^2 + Y^2\right) + m\,\omega\left(\dot{X}\,Y - \dot{Y}\,X\right)\right]$$

$$\frac{\partial L}{\partial Y} = \frac{\partial}{\partial Y}\left[\frac{m\,\omega^2}{2}\left(X^2 + Y^2\right) + m\,\omega\left(\dot{X}\,Y - \dot{Y}\,X\right)\right]$$

$$\frac{\partial L}{\partial Y} = \frac{m\,\omega^2}{2}\,\frac{\partial}{\partial Y}\left(X^2 + Y^2\right) + m\,\omega\,\frac{\partial}{\partial Y}\left(\dot{X}\,Y - \dot{Y}\,X\right)$$

$$\frac{\partial L}{\partial Y} = \frac{m\,\omega^2}{2}\,\frac{\partial}{\partial Y}\,X^2 + \frac{m\,\omega^2}{2}\,\frac{\partial}{\partial Y}\,Y^2 + m\,\omega\,\frac{\partial}{\partial Y}\,\dot{X}\,Y - m\,\omega\,\frac{\partial}{\partial Y}\,\dot{Y}\,X$$

$$\frac{\partial L}{\partial Y} = \frac{m\,\omega^2}{2}\,\frac{\partial}{\partial Y}\,Y^2 + m\,\omega\,\frac{\partial}{\partial Y}\,\dot{X}\,Y$$

$$\frac{\partial L}{\partial Y} = m\,\omega^2\,Y + m\,\omega\,\dot{X}$$

We then calculate $\partial L\big/\partial\dot{X}$

$$\frac{\partial L}{\partial \dot{X}} = \frac{\partial}{\partial \dot{X}}\left[\frac{m}{2}\left(\dot{X}^2 + \dot{Y}^2\right) + \frac{m\,\omega^2}{2}\left(X^2 + Y^2\right) + m\,\omega\left(\dot{X}\,Y - \dot{Y}\,X\right)\right]$$

We have one term that vanishes

$$\frac{\partial L}{\partial \dot{X}} = \frac{\partial}{\partial \dot{X}}\left[\frac{m}{2}\left(\dot{X}^2 + \dot{Y}^2\right) + m\,\omega\left(\dot{X}\,Y - \dot{Y}\,X\right)\right]$$

we can expand this using the sum rule,

$$\frac{\partial L}{\partial \dot{X}} = \frac{\partial}{\partial \dot{X}}\frac{m}{2}\left(\dot{X}^2 + \dot{Y}^2\right) + \frac{\partial}{\partial \dot{X}} m\,\omega\left(\dot{X}\,Y - \dot{Y}\,X\right)$$

then we use the constant multiple rule

$$\frac{\partial L}{\partial \dot{X}} = \frac{m}{2}\frac{\partial}{\partial \dot{X}}\left(\dot{X}^2 + \dot{Y}^2\right) + m\,\omega\frac{\partial}{\partial \dot{X}}\left(\dot{X}\,Y - \dot{Y}\,X\right)$$

we again use the sum rule

$$\frac{\partial L}{\partial \dot{X}} = \frac{m}{2}\frac{\partial}{\partial \dot{X}}\dot{X}^2 + \frac{m}{2}\frac{\partial}{\partial \dot{X}}\dot{Y}^2 + m\,\omega\frac{\partial}{\partial \dot{X}}\dot{X}\,Y - m\,\omega\frac{\partial}{\partial \dot{X}}\dot{Y}\,X$$

two terms vanish,

$$\frac{\partial L}{\partial \dot{X}} = \frac{m}{2}\frac{\partial}{\partial \dot{X}}\dot{X}^2 + m\,\omega\frac{\partial}{\partial \dot{X}}\dot{X}\,Y$$

so

$$\frac{\partial L}{\partial \dot{X}} = m\,\dot{X} + m\,\omega\,Y$$

We then calculate $\partial L \big/ \partial \dot{Y}$

$$\frac{\partial L}{\partial \dot{Y}} = \frac{\partial}{\partial \dot{Y}}\left[\frac{m}{2}\left(\dot{X}^2 + \dot{Y}^2\right) + \frac{m\,\omega^2}{2}\left(X^2 + Y^2\right) + m\,\omega\left(\dot{X}\,Y - \dot{Y}\,X\right)\right]$$

$$\frac{\partial L}{\partial \dot{Y}} = \frac{\partial}{\partial \dot{Y}}\left[\frac{m}{2}\left(\dot{X}^2 + \dot{Y}^2\right) + m\,\omega\left(\dot{X}\,Y - \dot{Y}\,X\right)\right]$$

$$\frac{\partial L}{\partial \dot{Y}} = \frac{\partial}{\partial \dot{Y}}\frac{m}{2}\left(\dot{X}^2 + \dot{Y}^2\right) + \frac{\partial}{\partial \dot{Y}} m\,\omega\left(\dot{X}\,Y - \dot{Y}\,X\right)$$

$$\frac{\partial L}{\partial \dot{Y}} = \frac{m}{2}\frac{\partial}{\partial \dot{Y}}\left(\dot{X}^2 + \dot{Y}^2\right) + m\,\omega\frac{\partial}{\partial \dot{Y}}\left(\dot{X}\,Y - \dot{Y}\,X\right)$$

$$\frac{\partial L}{\partial \dot{Y}} = \frac{m}{2}\frac{\partial}{\partial \dot{Y}}\dot{X}^2 + \frac{m}{2}\frac{\partial}{\partial \dot{Y}}\dot{Y}^2 + m\,\omega\frac{\partial}{\partial \dot{Y}}\dot{X}\,Y - m\,\omega\frac{\partial}{\partial \dot{Y}}\dot{Y}\,X$$

$$\frac{\partial L}{\partial \dot{Y}} = \frac{m}{2}\frac{\partial}{\partial \dot{Y}}\dot{Y}^2 - m\,\omega\frac{\partial}{\partial \dot{Y}}\dot{Y}\,X$$

$$\frac{\partial L}{\partial \dot{Y}} = m\,\dot{Y} - m\,\omega\,X$$

This give us the canonical momentum vector

$$\vec{p} = \left(m\,\dot{X} + m\,\omega\,Y,\ m\,\dot{Y} - m\,\omega\,X\right)$$

We take the time derivative of this to get the generalized force,

$$\frac{d}{dt}\vec{p} = \frac{d}{dt}\left(m\dot{X} + m\,\omega\,Y,\ m\dot{Y} - m\,\omega\,X\right)$$

$$\frac{d}{dt}\vec{p} = \left[\frac{d}{dt}\left(m\dot{X} + m\,\omega\,Y\right),\ \frac{d}{dt}\left(m\dot{Y} - m\,\omega\,X\right)\right]$$

$$\frac{d}{dt}\vec{p} = \left[\left(\frac{d}{dt}m\dot{X} + \frac{d}{dt}m\,\omega\,Y\right),\ \left(\frac{d}{dt}m\dot{Y} - \frac{d}{dt}m\,\omega\,X\right)\right]$$

$$\frac{d}{dt}\vec{p} = \left(m\ddot{X} + m\,\omega\,\dot{Y},\ m\ddot{Y} - m\,\omega\,\dot{X}\right)$$

We now write the two equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) = \frac{\partial L}{\partial X} \Rightarrow m\ddot{X} + m\,\omega\,\dot{Y} = m\,\omega^2\,X - m\,\omega\,\dot{Y} \Rightarrow \ddot{X} + \omega\,\dot{Y} = \omega^2\,X - \omega\,\dot{Y} \Rightarrow \ddot{X} = \omega^2\,X - 2\,\omega\,\dot{Y}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Y}}\right) = \frac{\partial L}{\partial Y} \Rightarrow m\ddot{Y} - m\,\omega\,\dot{X} = m\,\omega^2\,Y + m\,\omega\,\dot{X} \Rightarrow \ddot{Y} - \omega\,\dot{X} = \omega^2\,Y + \omega\,\dot{X} \Rightarrow \ddot{Y} = \omega^2\,Y + 2\,\omega\,\dot{X}$$