Lecture 6: Exercise 2

Explanation

This exercise is a many-degree-of-freedom version of Exercise 1.

Hint

Examine each component of the equations and determine what they mean. Be careful to note that we are summing over all degrees of freedom represented by the subscripts.

Answer

We begin be writing Equation (6),

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$$

We can rewrite this

$$\frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \dot{x}_i}\right) = \sum_i \frac{\partial L}{\partial x_i}$$

this is easy to deal with because of the sum rule

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1} + \frac{\partial L}{\partial \dot{x}_2} + \frac{\partial L}{\partial \dot{x}_3} + \ldots\right) = \frac{\partial L}{\partial x_1} + \frac{\partial L}{\partial x_2} + \frac{\partial L}{\partial x_3} + \ldots$$

Since

$$L = T + V = \frac{1}{2} m \dot{x}_i^2 + V(x_i)$$

then

$$\frac{\partial L}{\partial x_i} = \frac{\partial}{\partial x_i} V(x_i)$$

from the definition of force, this is F_i from Newton's equation of motion. Then

$$\frac{\partial L}{\partial \dot{x}_i} = 2 \times \frac{1}{2} m \dot{x}_i = m \dot{x}_i = p_i$$

this is the canonical momentum. Taking its time derivative,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{d}{dt} m \dot{x}_i = m \ddot{x}_i = m a_i$$

so, we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i} \Longrightarrow m a_i = F_i$$

thus Lagrange's equations and Newton's equation are, once again, the same.