Lecture 5: Exercise 3

Explanation

This exercise is designed to give more experience in using potentials, and developing a physical intuition about them. I will introduce the notion of an effective potential, along the way I will derive the angular momentum.

Hint

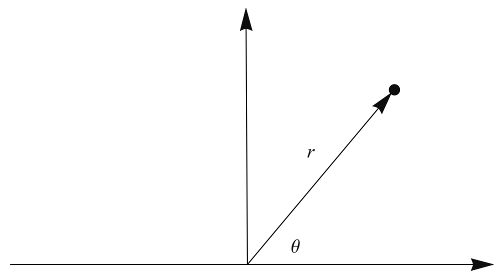
This requires thinking carefully about the nature of a circular orbit. You need to realize that for a circular orbit we can treat the motion as a one dimensional motion around the circle. The potential must have a minimum at the radius of the orbit for such stable orbits to exist.

Answer

To begin with, lets look at writing Newton's equation of motion in polar coordinates.

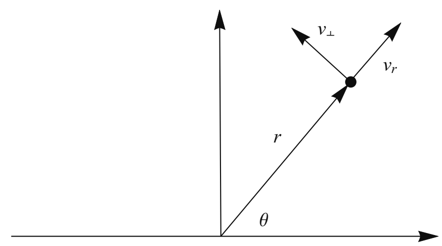
$$\vec{F} = m\,\vec{a}$$

To do this we need to find the components of acceleration in polar coordinates. Unlike Cartesian coordinates, we have components in the $(r,\ \theta)$ directions instead of the $(x,\ y)$ directions.



A particle in polar coordinates.

We begin with the velocity. What is v in polar coordinates? To proceed in this way we can set up a local coordinate system that looks Cartesian, with one axis in the r direction and the other in the perpendicular, $\perp$, direction.

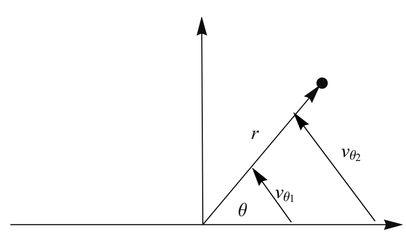


Inventing a local coordinate system.

The velocity vector now has one component in the r direction and one in the $\perp$ direction. The r component of velocity is, not surprisingly,

$$v_r = \dot{r}.$$

What about the velocity in the perpendicular direction? You can see this if you hold out your arm and bend your elbow, your fingers will move faster than your elbow.



We can see that close in to your elbow the velocity is simply how much of an angle you traverse in a given unit of time, $\dot{\theta}$. Further up your arm we must multiply the distance away from the origin, r,

$$v_\perp = r\,\dot{\theta}.$$

where $\dot{\theta}$ is the angular velocity, the change in angle per unit time. So we can write the velocity vector

$$\vec{v} = \dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta}$$

Now we need to find the acceleration vector,

$$\vec{a} = \frac{d\,\vec{v}}{d\,t} = \frac{d}{d\,t}\dot{r}\,\hat{r} + \frac{d}{d\,t}r\,\dot{\theta}\,\hat{\theta}.$$

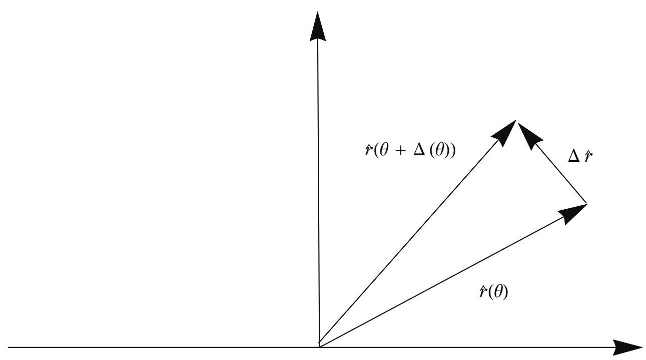
Again we have a component in the radial direction and the perpendicular direction. Let’s take the radial direction first.

$$\frac{d}{dt}\dot{r}\,\hat{r} = \dot{r}\frac{d}{dt}\hat{r} + \hat{r}\frac{d}{dt}\dot{r} = \dot{r}\frac{d}{dt}\hat{r} + \hat{r}\,\ddot{r}$$

We can use the chain rule for the first term, and if we think about it long enough it will occur to us to try using the angular velocity,

$$\frac{d\hat{r}}{dt} = \frac{d\hat{r}}{d\theta}\frac{d\theta}{dt} = \frac{d\hat{r}}{d\theta}\dot{\theta}$$

What does $\frac{d\hat{r}}{d\theta}$ mean? Since this is a unit vector it has a constant magnitude. There is a principle of vector differentiation that tells us the derivative of a constant vector is perpendicular to the vector. In this case the derivative of the radial unit vector is in the direction of the perpendicular vector. We have the direction of $d\hat{r}/dt$, but we need the magnitude. This is the length of the vector, so we have this situation,



These three vectors form an isosceles triangle. The vector $\Delta\hat{r}$ is then

$$\sin\Delta\theta = \frac{\Delta\hat{r}}{\hat{r}(\theta)}$$

or

$$\Delta\hat{r} = \hat{r}(\theta)\sin\Delta\theta$$

the magnitude of this is

$$\left|\Delta\hat{r}\right| = \sqrt{\hat{r}(\theta)\sin\Delta\theta\cdot\hat{r}(\theta)\sin\Delta\theta} = \hat{r}(\theta)\sin\Delta\theta$$

so, as $\Delta\theta \to 0$ the sine gets approximately equal to $\Delta\theta$. Now, we already have the fact that $\hat{r}(\theta) = 1$

$$\Delta\hat{r} = \Delta\theta$$

so we are led to the conclusion that

$$\frac{d\hat{r}}{d\theta} = \hat{\theta}.$$

So,

$$\frac{d\hat{r}}{dt} = \frac{d\hat{r}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\hat{\theta}$$

Thus

$$\frac{d}{dt}\dot{r}\,\hat{r} = \dot{r}\frac{d}{dt}\hat{r} + \hat{r}\frac{d}{dt}\dot{r} = \dot{r}\frac{d}{dt}\hat{r} + \hat{r}\,\ddot{r} = \dot{r}\,\dot{\theta}\,\hat{\theta} + \hat{r}\,\ddot{r}.$$

We now turn to the perpendicular component,

$$\frac{d}{dt}r\,\dot{\theta}\,\hat{\theta} = \dot{\theta}\,\hat{\theta}\frac{d}{dt}r + r\,\hat{\theta}\frac{d}{dt}\dot{\theta} + r\,\dot{\theta}\frac{d}{dt}\hat{\theta} = \dot{r}\,\dot{\theta}\,\hat{\theta} + r\,\ddot{\theta}\,\hat{\theta} + r\,\dot{\theta}\frac{d}{dt}\hat{\theta}$$

now we have another derivative of a unit vector. We can perform the same sort of analysis, leading us to the fact that the derivative is in the radial direction (perpendicular to θ). we can use the chain rule again to get

$$r\dot{\theta}\frac{d}{dt}\theta = r\dot{\theta}\frac{d\hat{\theta}}{d\theta}\frac{d\theta}{dt} = r\dot{\theta}^2\frac{d\hat{\theta}}{d\theta}$$

Its magnitude can be found by examining some dot products. Since $\hat{r}$ and $\hat{\theta}$ are perpendicular, then

$$\hat{r}\cdot\hat{\theta} = 0$$

so, if we take the derivative of this

$$\frac{d}{d\theta}\left(\hat{r}\cdot\hat{\theta}\right) = 0$$

the derivative of the dot product is also governed by the product rule,

$$\frac{d}{d\theta}\left(\hat{r}\cdot\hat{\theta}\right) = \theta\cdot\frac{d\hat{r}}{d\theta} + \hat{r}\cdot\frac{d\hat{\theta}}{d\theta}$$

we already know that $\frac{d\hat{r}}{d\theta} = \hat{\theta}$, so

$$\frac{d}{d\theta}\left(\hat{r}\cdot\hat{\theta}\right) = \theta\cdot\theta + \hat{r}\cdot\frac{d\hat{\theta}}{d\theta}$$

since these are unit vectors $\hat{\theta}\cdot\hat{\theta} = 1$

$$\frac{d}{d\theta}\left(\hat{r}\cdot\hat{\theta}\right) = 1 + \hat{r}\cdot\frac{d\hat{\theta}}{d\theta}$$

since $\frac{d}{d\theta}\left(\hat{r}\cdot\hat{\theta}\right) = 0$, we have

$$0 = 1 + \hat{r}\cdot\frac{d\hat{\theta}}{d\theta}$$

or

$$\hat{r}\cdot\frac{d\hat{\theta}}{d\theta} = -1$$

This is an interesting expression. If you look at it long enough it tells you that the dot product is not zero. This means that $\hat{r}$ and $\frac{d\hat{\theta}}{d\theta}$ are not perpendicular. In fact, since they are unit vectors, then they are not exactly equal, or their dot product would be equal to 1. Since it is -1, then they are equal, but opposite, so

$$\frac{d\hat{\theta}}{d\theta} = -\hat{r}.$$

Thus

$$\begin{aligned}\frac{d}{dt}r\dot{\theta}\hat{\theta} &= \dot{\theta}\hat{\theta}\frac{d}{dt}r + r\hat{\theta}\frac{d}{dt}\dot{\theta} + r\dot{\theta}^2\frac{d\hat{\theta}}{d\theta}\\ &= \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r}\\ &= \left(\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\theta} - r\dot{\theta}^2\hat{r}\end{aligned}$$

and the acceleration is now

$$\begin{aligned}\vec{a} = \frac{d\vec{v}}{dt} &= \frac{d}{dt}\dot{r}\hat{r} + \frac{d}{dt}r\dot{\theta}\hat{\theta}\\ &= \dot{r}\dot{\theta}\hat{\theta} + \hat{r}\ddot{r} + \left(\hat{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\theta} - r\dot{\theta}^2\hat{r}\\ &= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(\dot{r}\dot{\theta} + \dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\theta}\\ &= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\theta}\end{aligned}$$

so we have our expression for acceleration. This lays some of the ground work for what follows. We can use this to write Newton's equations of motion in polar coordinates.

$$\vec{F} = m\,\vec{a} = m\left[\left(\ddot{r} - r\,\dot{\theta}^2\right)\hat{r} + \left(2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta}\right)\hat{\theta}\right]$$

We know that the force of gravity depends only on the distance from some large mass. So we can rewrite the force $F(r)\,\hat{r}$,

$$F(r)\,\hat{r} = m\left[\left(\ddot{r} - r\,\dot{\theta}^2\right)\hat{r} + \left(2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta}\right)\hat{\theta}\right]$$

This gives us two different differential equations to solve. One for the radial component and one for the perpendicular component.

$$F(r) = m\,\left(\ddot{r} - r\,\dot{\theta}^2\right)$$

$$0 = m\left(2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta}\right)$$

this second one can be rewritten

$$0 = 2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta}$$

It turns out that we can solve this independently of the first equation. Now there is a technique for solving differential equations by changing their forms by introducing a factor, thus producing a derivative that we can then integrate. This is called an *integrating factor*. If we look at this equation something nags at us a little bit. It almost looks like a product rule, so let's try using r as our integrating factor,

$$0 = 2\,r\,\dot{r}\,\dot{\theta} + r^2\,\ddot{\theta}$$

it is a product rule,

$$\frac{d}{d\,t}\,r^2\,\dot{\theta} = 2\,r\,\dot{r}\,\dot{\theta} + r^2\,\ddot{\theta}$$

and we know this is equal to 0,

$$\frac{d}{d\,t}\,r^2\,\dot{\theta} = 0,$$

so $r^2\,\dot{\theta}$ is a constant. It turns out that this is an important quantity—it is the angular momentum per unit mass. Angular momentum can be though of as the momentum in the perpendicular direction. We can write the angular momentum as L and the angular momentum per unit mass as l,

$$l = \frac{L}{m} = r^2\,\dot{\theta}$$

or

$$\dot{\theta} = \frac{l}{r^2}.$$

So we have one of the equations that determine an orbit. The other would be the radial equation. There are a couple of ways to get that one, we use the discussion of energy. We know that the total energy is the sum of of the potential and kinetic energies:

$$E = T\; +\; V$$

where the kinetic energy for polar coordinates is

$$\begin{aligned}
T &= \frac{1}{2}\,m\,v^2 \\
&= \frac{1}{2}\,m\left(\dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta}\right)^2 \\
&= \frac{1}{2}\,m\left(\dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta}\right)\cdot\left(\dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta}\right) \\
&= \frac{1}{2}\,m\,\dot{r}^2 + \frac{1}{2}\,m\,r^2\,\dot{\theta}^2 \\
&= \frac{1}{2}\,m\,\dot{r}^2 + \frac{1}{2}\,m\,r^2\,\frac{l^2}{r^4}
\end{aligned}$$

$$= \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m \frac{l^2}{r^2}$$

$$= \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m \frac{L^2}{m^2 r^2}$$

$$= \frac{1}{2} m \dot{r}^2 + \frac{L^2}{2 m r^2}$$

so,

$$E = \frac{1}{2} m \dot{r}^2 + \frac{L^2}{2 m r^2} + V$$

we can rewrite this

$$E - \frac{1}{2} m \dot{r}^2 - \frac{L^2}{2 m r^2} - V = 0$$

$$E - \frac{L^2}{2 m r^2} - V = \frac{1}{2} m \dot{r}^2$$

$$\frac{2}{m}\left(E - \frac{L^2}{2 m r^2} - V\right) = \dot{r}^2$$

or,

$$\dot{r} = \pm \sqrt{\frac{2}{m}\left(E - \frac{L^2}{2 m r^2} - V\right)}$$

if we look at this long enough we can make a simplification, we can introduce a new variable

$$U = V + \frac{L^2}{2 m r^2}$$

what we call the *effective potential*. Then

$$\dot{r} = \pm \sqrt{\frac{2}{m}(E - U)}$$

For our potential, we have

$$V = \frac{k}{2\left(x^2 + y^2\right)}$$

where we can write

$$r^2 = x^2 + y^2$$

so

$$V = \frac{k}{2 r^2}$$

the effective potential is then

$$U = \frac{k}{2 r^2} + \frac{L^2}{2 m r^2}$$

Now we have something interesting. For a stable circular orbit to exist, the effective potential must have a minimum value at the orbital radius r_c. Why is this required? If we think about it a circle is defined as a figure consisting of all points a fixed distance from the center of the circle. Having a fixed radius, the circle is stable. The only way we can have motion around this circle is if the potential is stationary at the radius of the circle. Since motion is confined to a circular path we can think of it as one-dimensinal motion, as if we unwarapped the circle and made an infinite line out of it. If the potential were a maximum then the motion would be forced away from the circle. Since it is a minimum the motion is forced into the circle. So, for the circular radius we have the effective potential,

$$U(r_c) = \frac{k}{2\,r_c^2} + \frac{L^2}{2\,m\,r_c^2}$$

to find a minimum we need a stationary point

$$U'(r_c) = \frac{d}{d\,r_c}\left(\frac{k}{2\,r_c^2} + \frac{L^2}{2\,m\,r_c^2}\right) = -\frac{2\,k}{2\,r_c^3} - \frac{2\,L^2}{2\,m\,r_c^3} = -\frac{k}{r_c^3} - \frac{L^2}{m\,r_c^3} = 0$$

or

$$-\frac{L^2}{r_c^3} = \frac{k\,m}{r_c^3}$$

or

$$k\,m = -L^2$$

and, we also need the second derivative to be positive,

$$U''(r_c) = -\frac{d}{d\,r_c}\left(\frac{k}{r_c^3} + \frac{L^2}{m\,r_c^3}\right) = \frac{3\,k}{r_c^4} + \frac{3\,L^2}{m\,r_c^4} > 0$$

or

$$\frac{3\,k}{r_c^4} > -\frac{3\,L^2}{m\,r_c^4}$$

$$\frac{3\,k\,m}{r_c^4} > -\frac{3\,L^2}{r_c^4}$$

$$3\,k\,m > -3\,L^2$$

$$k\,m > -L^2$$

This result is puzzling. What we have done is assumed that there can be a circular orbit. By the requirement of a stationary point we have the condition $k\,m = -L^2$, and by the requirement of a minimum value we have the condition $k\,m > -L^2$. Since no contradiction can ever be true, by the rules of logic, then we have shown that there are no stable circular orbits for this potential. Note also that we did this by thinking about the conditions for a circular orbit and never had to solve the equations of motion. By the way, this method of demonstrating a contradiction to prove the opposite of the case you are examining is called *proof by contradiction*, or *reductio ad absurdum*.