Lecture 5: Exercise 2

Explanation

This exercise has several goals:

1. Use the idea of a gradient in the kind of problem you might actually run into.
2. Derive an equation of motion given a potential.
3. Deduce the properties of trajectories based on the equation of motion.
4. Demonstrate the law of conservation of energy.

Hint

Recall the definition of the gradient, conservative forces, $\vec{F} = -\nabla V$, and Newton's second law of motion.

Answer

We have

$$V = \frac{1}{2} k \left(x^2 + y^2\right)$$

we can write

$$r^2 = x^2 + y^2$$

so

$$V = \frac{1}{2} k\, r^2$$

we calculate the gradient:

$$\nabla\ V = \frac{\partial V}{\partial r} = k\, \vec{r}$$

Newton’s second law of motion is

$$\vec{F} = m\, \ddot{\vec{r}}.$$

Since

$$\vec{F} = -\nabla\ V = -k\, \vec{r}$$

we have

$$-k\, \vec{r} = m\, \ddot{\vec{r}}.$$

This is the equation of motion for a harmonic oscillator. But notice that this is also a vector equation, in this case it means that the equation has the same form no matter what coordiantes system we use, and no matter what direction we are looking. When something is independent of direction we call it *isotropic*. So we have derived the equation of motion for an isotropic oscillator. We can rewrite this,

$$-\frac{k}{m}\, \vec{r} = \ddot{\vec{r}}$$

and make thhe substitution $\omega = k / m$, then we write

$$-\omega\, \vec{r} = \ddot{\vec{r}}$$

The general solution of this equation is

$$\vec{r}(t) = \vec{c}_1 \cos \omega\, t\ + \vec{c}_2 \sin \omega\, t$$

where $\vec{c}_1$ and $\vec{c}_2$ are constant vectors that we can determine by examining the initial conditions of the problem. If we choose our initial time to be $t = 0$, then we can assume that $\vec{r}(0) = \vec{r}_0$ is the initial position and the initial velocity is $\vec{v}_0$, then

$$\vec{r}(0) = \vec{c}_1 \cos 0\ + \vec{c}_2 \sin 0 = \vec{c}_1 = \vec{r}_0$$

and the derivative is

$$\dot{\vec{r}}(t) = \vec{c}_2\, \omega \cos \omega\, t - \vec{c}_1\, \omega \sin \omega\, t$$

or

$$\dot{\vec{r}}(0) = \vec{c}_2\, \omega \cos 0 - \vec{c}_1\, \omega \sin 0 = \vec{c}_2\, \omega$$

another way of writing this is

$$\frac{\dot{\vec{r}}(0)}{\omega} = \vec{c}_2 = \frac{\vec{v}_0}{\omega}$$

we can substitute these back into the general solution to get

$$\vec{r}(t) = \vec{r}_0 \cos \omega\, t\ + \frac{\vec{v}_0}{\omega} \sin \omega\, t$$

To find out the shape of any orbit we can rewrite this for a given angle θ

$$\vec{r}(t) = \vec{b}_1 \cos (\omega\, t - \theta)\ + \vec{b}_2 \sin\ (\omega\, t - \theta)$$

where we have

$$\vec{b}_1 = \vec{c}_1 \cos\theta + \vec{c}_2 \sin\theta$$

and

$$\vec{b}_2 = \vec{c}_2 \cos\theta - \vec{c}_1 \sin\theta$$

If we make the assumption that $\vec{b}_1$ and $\vec{b}_2$ are perpendicular then the dot product vanishes,

$$\vec{b}_1 \cdot \vec{b}_2 = -\left(\vec{c}_1^{\,2} - \vec{c}_2^{\,2}\right) \sin\theta \cos\theta + \vec{c}_1 \cdot \vec{c}_2 \left(\cos^2\theta - \sin^2\theta\right) = 0.$$

or

$$\vec{c}_1 \cdot \vec{c}_2 \left(\cos^2\theta - \sin^2\theta\right) = -\left(\vec{c}_1^{\,2} - \vec{c}_2^{\,2}\right) \sin\theta \cos\theta$$

or

$$\vec{c}_1 \cdot \vec{c}_2 = -\left(\vec{c}_1^{\,2} - \vec{c}_2^{\,2}\right) \frac{\sin\theta \cos\theta}{\cos^2\theta - \sin^2\theta}$$

and

$$\frac{\vec{c}_1 \cdot \vec{c}_2}{\vec{c}_1^{\,2} - \vec{c}_2^{\,2}} = -\frac{\sin\theta \cos\theta}{\cos^2\theta - \sin^2\theta}.$$

We use the identity $\cos^2\theta - \sin^2\theta = \cos 2\theta$,

$$\frac{\vec{c}_1 \cdot \vec{c}_2}{\vec{c}_1^{\,2} - \vec{c}_2^{\,2}} = -\frac{\sin\theta \cos\theta}{\cos 2\theta}$$

this leads to another trigonometric identity $\sin 2\theta = 2 \sin\theta \cos\theta$, which we can manipulate a little to get $\sin 2\theta / 2 = \sin\theta \cos\theta$, so

$$\frac{\vec{c}_1 \cdot \vec{c}_2}{\vec{c}_1^{\,2} - \vec{c}_2^{\,2}} = -\frac{1}{2} \frac{\sin 2\theta}{\cos 2\theta}.$$

then we have, by definition,

$$\frac{\vec{c}_1 \cdot \vec{c}_2}{\vec{c}_1^{\,2} - \vec{c}_2^{\,2}} = -\frac{1}{2} \tan 2\theta\,.$$

Since we can always choose the coordiante axes of our system, we can write it so that x is in the direction of $\vec{b}_1$ and y is in the direction of $\vec{b}_2$,

$$x = b_1 \cos(\omega t - \theta)$$

$$y = b_2 \sin(\omega t - \theta)$$

$$z = 0$$

This shows us that the orbit is in a plane. If we apply the Pythagorean theorem to find the distance to a point from the origin,

$$\text{distance}^2 = x^2 + y^2 + z^2$$

$$\text{distance}^2 = [b_1 \cos(\omega t - \theta)]^2 + [b_2 \sin(\omega t - \theta)]^2$$

$$\text{distance}^2 = b_1^{\,2} \cos^2(\omega t - \theta) + b_2^2 \sin^2(\omega t - \theta)$$

where we to be able to eliminate the constants we would have.

$$\cos^2(\omega t - \theta) + \sin^2(\omega t - \theta) = 1$$

so we can do this by writing

$$\frac{x^2}{b_1^{\,2}} + \frac{y^2}{b_2^{\,2}} = 1.$$

This is the equation of an ellipse. If $b_1 = b_2$ then we have a circular orbit, if $b_1 = b_2 = 1$ then we have a circular orbit of radius 1.