Lecture 2: Exercise 8

Explanation

This is a straightforward calculation coupled with a visualization. The goal here is to see the relationship between position vectors, velocity vectors, and acceleration vectors.

Hint

Recall the definitions of velocity and acceleration.

Answer

- $\vec{r} = (\cos \omega t, e^{\omega t})$

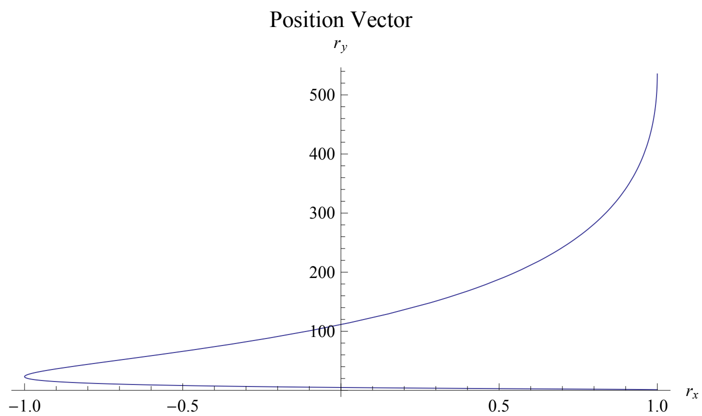
The first task is to calculate the derivative of each component with respect to time, thus giving us the velocity vector:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(\cos \omega t, e^{\omega t}) = \left(\frac{d}{dt}\cos \omega t, \frac{d}{dt}e^{\omega t}\right) = (-\omega \sin \omega t, \omega e^{\omega t}).$$

We then calculate the acceleration vector

$$\vec{a} = \frac{d\,\vec{v}}{d\,t} = \frac{d}{d\,t}\left(-\omega \sin \omega\, t,\ \omega\, e^{\omega\, t}\right)$$
$$= \left(-\frac{d}{d\,t}\,\omega \sin \omega\, t,\ \frac{d}{d\,t}\,\omega\, e^{\omega\, t}\right)$$
$$= \left(-\omega^2 \cos \omega\, t,\ \omega^2\, e^{\omega\, t}\right).$$

Out[13]=



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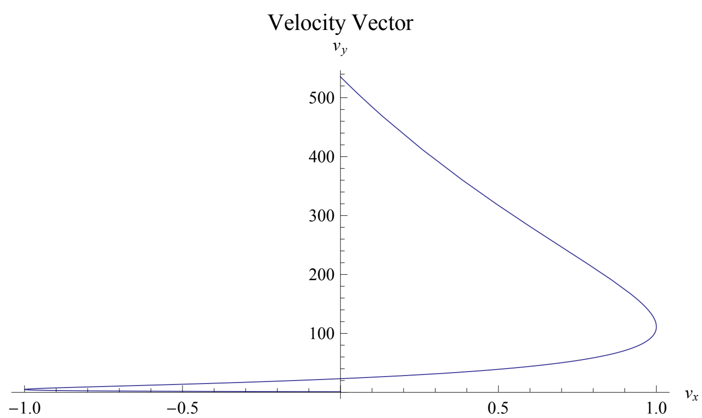

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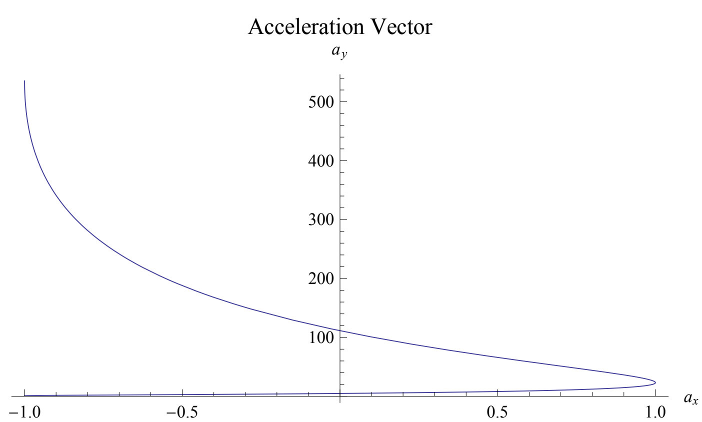
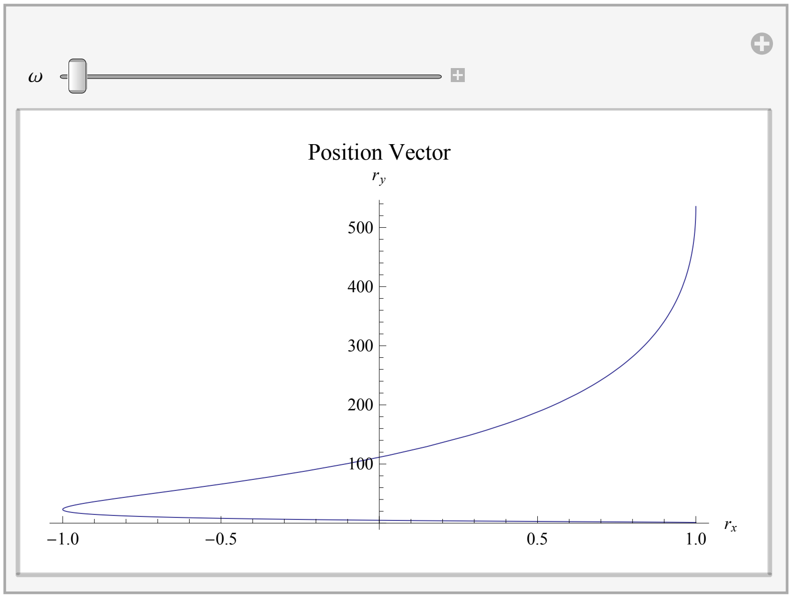


If you are viewing this with the CDF Player, you can manipulate the value of ω by moving the slider—watch the vertical axis scale change as you increase ω.

Out[18]=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Position Vector
r_y
500
400
300
200
100
r_x
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−0.5
0.5
1.0

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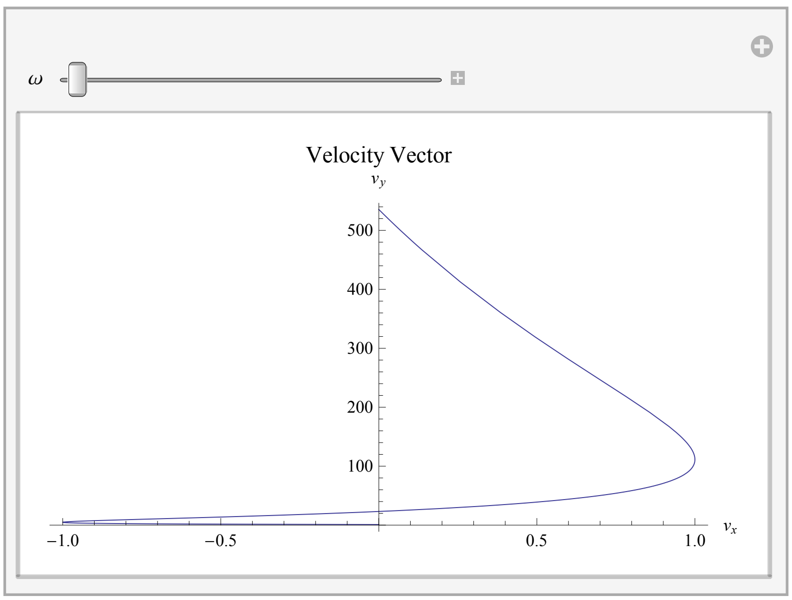
ω
Velocity Vector
v_y
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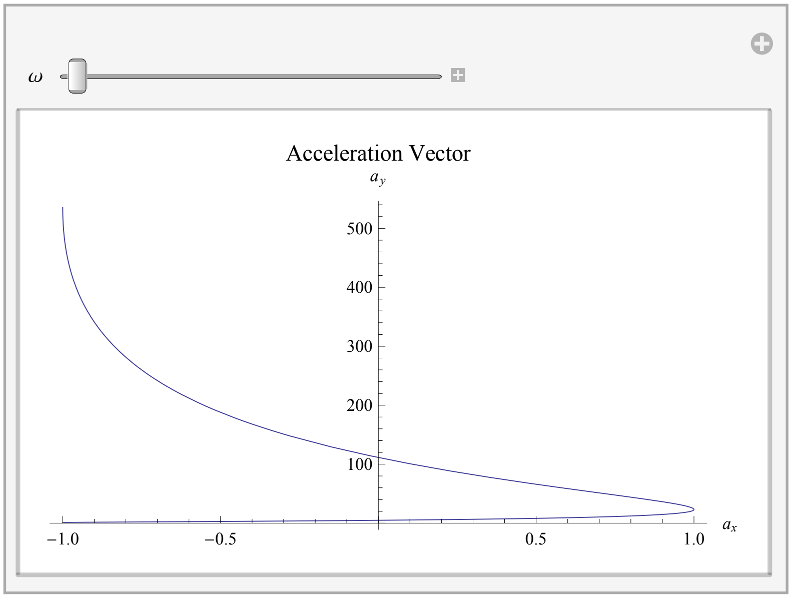
400
300
200
100
v_x
−1.0
−0.5
0.5
1.0

Out[20]=



- $\vec{r} = (\cos(\omega t - \phi), \sin(\omega t - \phi))$

The first task is to calculate the derivative of each component with respect to time, thus giving us the velocity vector:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}[\cos(\omega t - \phi),\ \sin(\omega t - \phi)] = \left[\frac{d}{dt}\cos(\omega t - \phi),\ \frac{d}{dt}\sin(\omega t - \phi)\right]$$

We use the trigonometric identities from Interlude 1:

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta,$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

Then we have:

$$\vec{v} = \left[\frac{d}{dt}(\cos\omega t\cos\phi - \sin\omega t\sin\phi),\ \frac{d}{dt}(\sin\omega t\cos\phi - \cos\omega t\sin\phi)\right]$$

$$= \left(\frac{d}{dt}\cos\omega t\cos\phi - \frac{d}{dt}\sin\omega t\sin\phi,\ \frac{d}{dt}\sin\omega t\cos\phi - \frac{d}{dt}\cos\omega t\sin\phi\right)$$

We then apply the product rule:

$$\vec{v}$$

$$\vec{v} = \left(\cos\phi\frac{d}{dt}\cos\omega t + \cos\omega t\frac{d}{dt}\cos\phi - \sin\phi\frac{d}{dt}\sin\omega t - \sin\omega t\frac{d}{dt}\sin\phi,\right.$$
$$\left.\cos\phi\frac{d}{dt}\sin\omega t + \sin\omega t\frac{d}{dt}\cos\phi - \sin\phi\frac{d}{dt}\cos\omega t - \cos\omega t\frac{d}{dt}\sin\phi\right)$$

Now, $\cos\phi$ is a constant,

$$\vec{v} = \left(\cos\phi\frac{d}{dt}\cos\omega t - \sin\phi\frac{d}{dt}\sin\omega t,\ \cos\phi\frac{d}{dt}\sin\omega t - \sin\phi\frac{d}{dt}\cos\omega t\right) =$$
$$(-\omega\cos\phi\sin\omega t - \omega\sin\phi\cos\omega t,\ \omega\cos\phi\cos\omega t + \omega\sin\phi\sin\omega t)$$

We now reverse the trigonometric identities:

$$\vec{v} = [-\omega\sin(\omega t - \phi),\ \omega\cos(\omega t - \phi)]$$

We then calculate the acceleration vector

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}[-\omega \sin(\omega t - \phi),\ \omega \cos(\omega t - \phi)] = \left[-\frac{d}{dt}\omega \sin(\omega t - \phi),\ \frac{d}{dt}\omega \cos(\omega t - \phi)\right]$$

we then apply the trigonometric identites:

$$\vec{a} = \left[-\omega \frac{d}{dt}(\sin\omega t \cos\phi - \cos\omega t \sin\phi),\ \omega \frac{d}{dt}(\cos\omega t \cos\phi - \sin\omega t \sin\phi),\right]$$

$$= \left(-\omega \frac{d}{dt}\sin\omega t \cos\phi + \omega \frac{d}{dt}\cos\omega t \sin\phi,\ \omega \frac{d}{dt}\cos\omega t \cos\phi - \omega \frac{d}{dt}\sin\omega t \sin\phi\right)$$

and then we apply the product rule

$$\vec{a} = \left(-\omega \sin\omega t \frac{d}{dt}\cos\phi - \omega \cos\phi \frac{d}{dt}\sin\omega t + \omega \cos\omega t \frac{d}{dt}\sin\phi + \omega \sin\phi \frac{d}{dt}\cos\omega t,\right.$$

$$\left.\omega \cos\omega t \frac{d}{dt}\cos\phi + \omega \cos\phi \frac{d}{dt}\cos\omega t - \omega \sin\omega t \frac{d}{dt}\sin\phi - \omega \sin\phi \frac{d}{dt}\sin\omega t\right)$$

$$= \left(-\omega \cos\phi \frac{d}{dt}\sin\omega t + \omega \sin\phi \frac{d}{dt}\cos\omega t,\ \omega \cos\phi \frac{d}{dt}\cos\omega t - \omega \sin\phi \frac{d}{dt}\sin\omega t\right)$$

$$= \left(-\omega^2 \cos\phi \cos\omega t - \omega^2 \sin\phi \sin\omega t,\ -\omega^2 \cos\phi \sin\omega t - \omega^2 \sin\phi \cos\omega t\right)$$

we reverse the trigonometric identities

$$\vec{a} = \left[-\omega^2 \cos(\omega t - \phi),\ -\omega^2 \sin(\omega t - \phi)\right].$$

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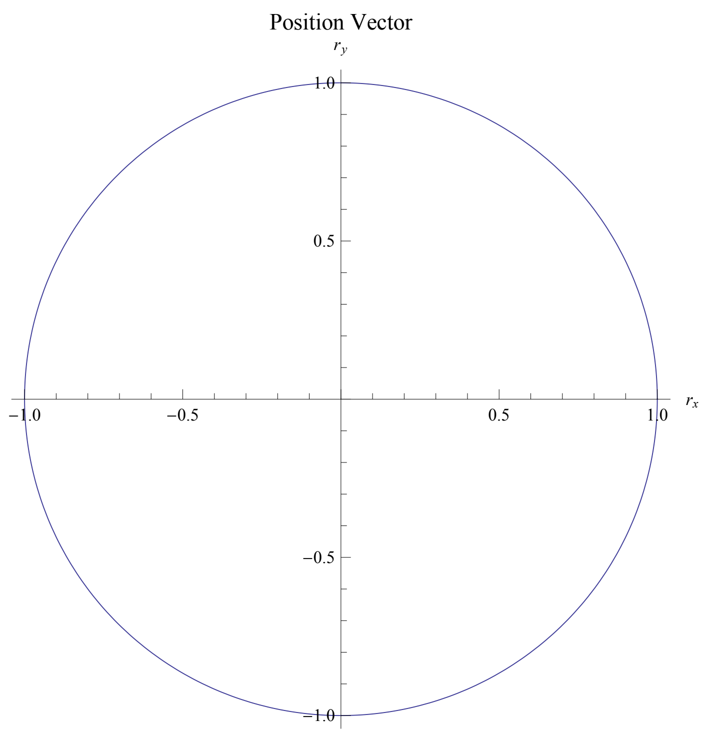

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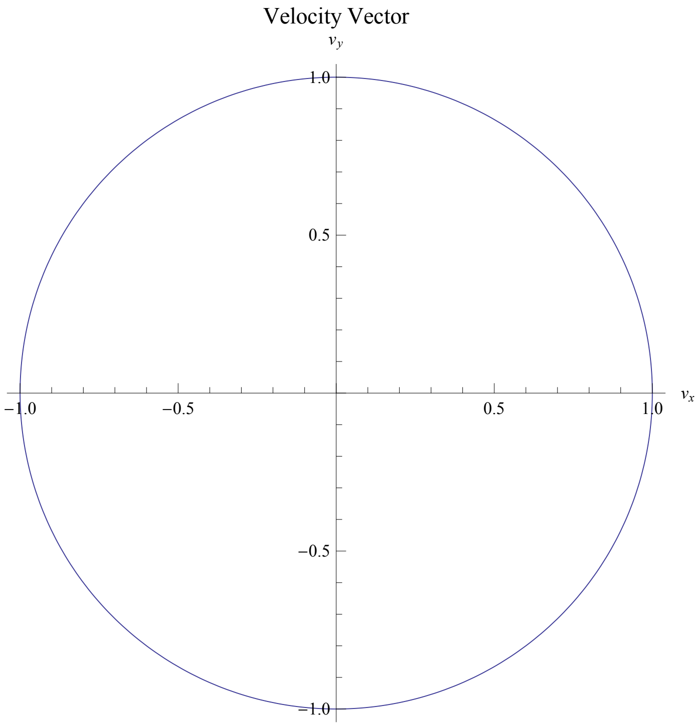

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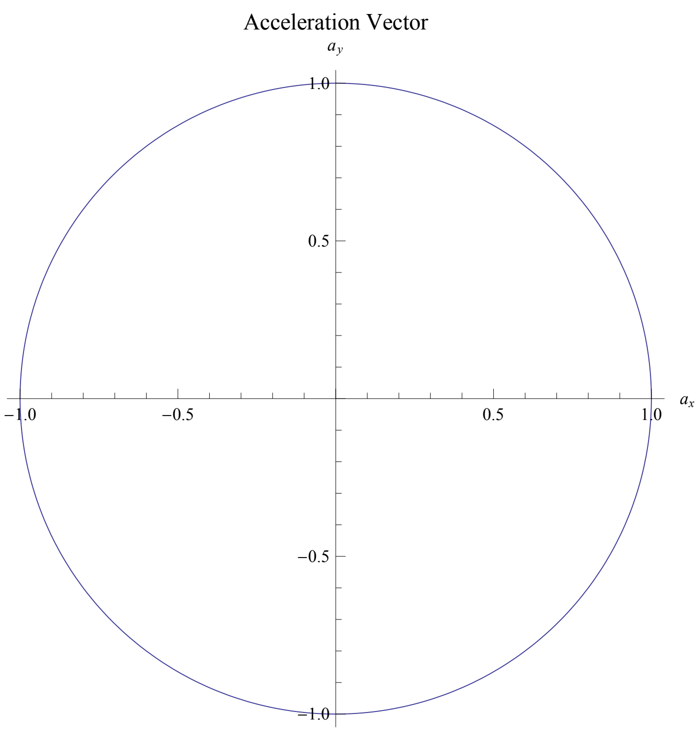
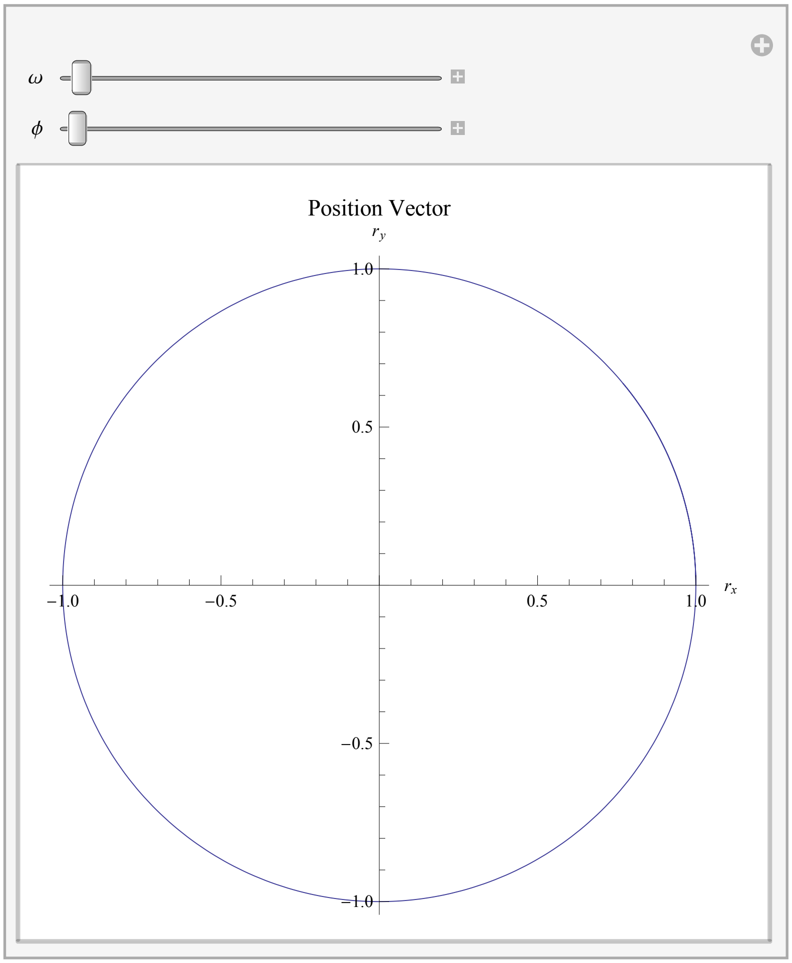


If you are viewing this with the CDF Player, you can manipulate the value of ω by moving the slider—watch the vertical axis scale change as you increase ω and/or ϕ,

Out[34]=



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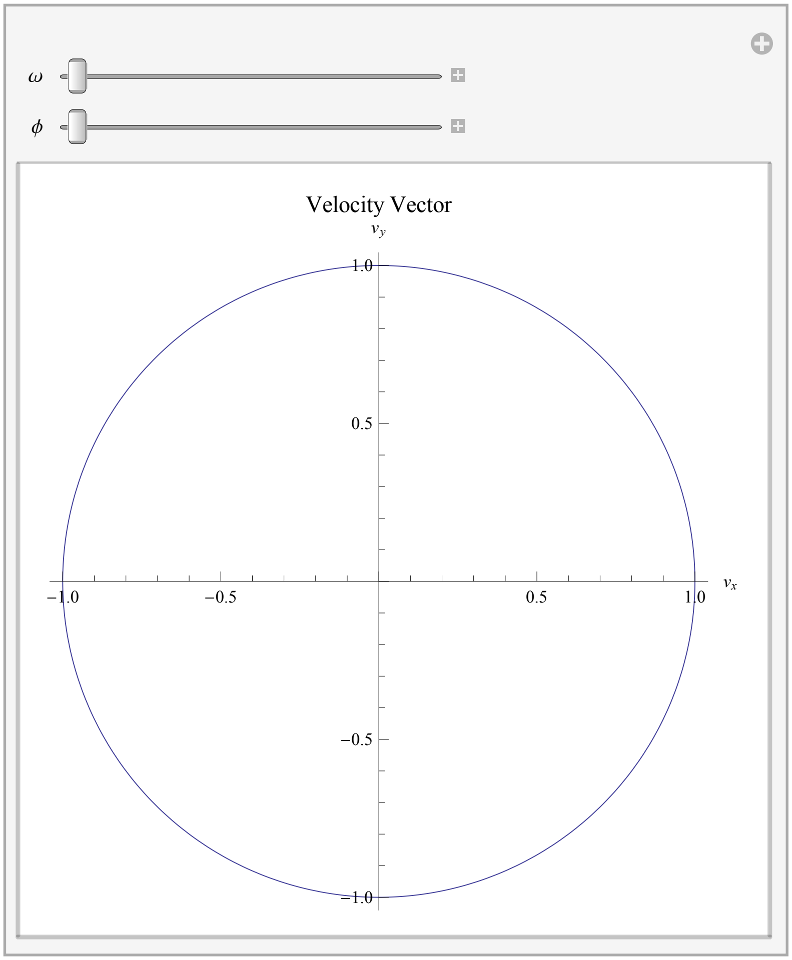

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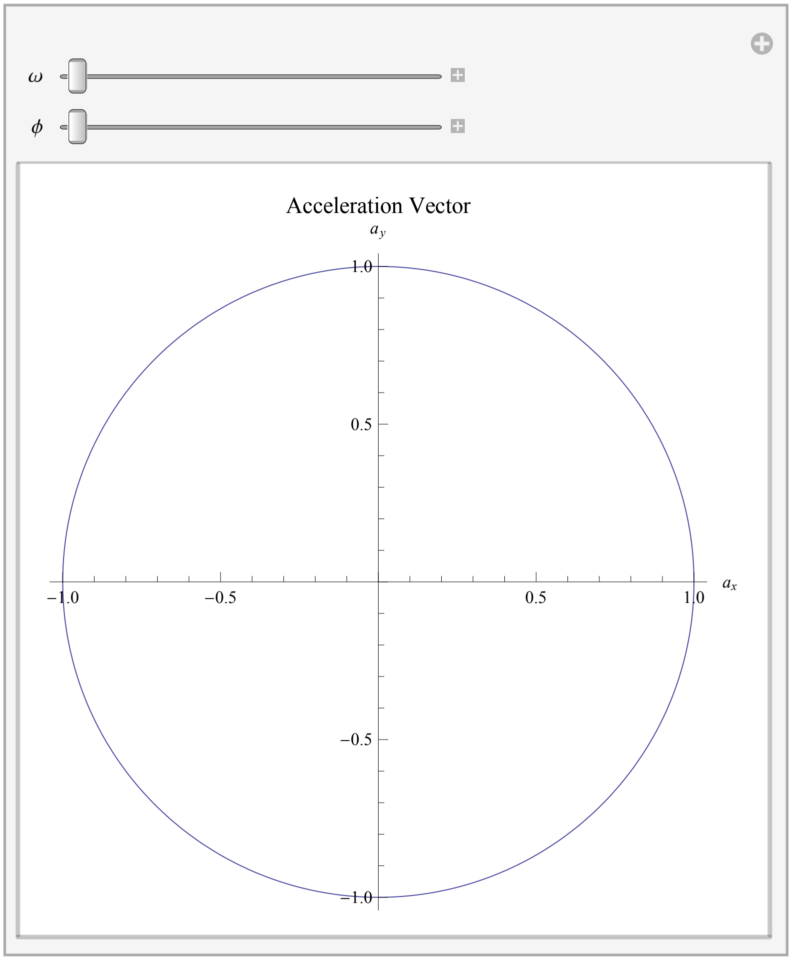


- $\vec{r} = (c\cos^3 t,\ c\sin^3 t)$

The first task is to calculate the derivative of each component with respect to time, thus giving us the velocity vector:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(c\cos^3 t,\ c\sin^3 t) = \left(\frac{d}{dt}c\cos^3 t,\ \frac{d}{dt}c\sin^3 t\right).$$

The easiest way to do this is by the chain rule, where we choose the variables $g = \cos t$, and $h = \sin t$. This gives us,

$$\frac{d}{dg}(c\,g^3) = 3\,c\,g^2;\ \frac{d}{dh}(c\,h^3) = 3\,c\,h^2$$

where

$$\frac{dg}{dt} = -\sin t;\ \frac{dh}{dt} = \cos t$$

so,

$$\vec{v} = \left(\frac{d}{dg}(c\,g^3)\frac{dg}{dt},\ \frac{d}{dh}(c\,h^3)\frac{dh}{dt}\right) = \left(3\,c\,g^2\frac{dg}{dt},\ 3\,c\,h^2\frac{dh}{dt}\right) = (-3\,c\,g^2\sin t,\ 3\,c\,h^2\cos t) = (-3\,c\cos^2 t\sin t,\ 3\,c\sin^2 t\cos t)$$

We then calculate the acceleration vector in a similar way,

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(-3\,c\cos^2 t\sin t,\ 3\,c\sin^2 t\cos t)$$

where,

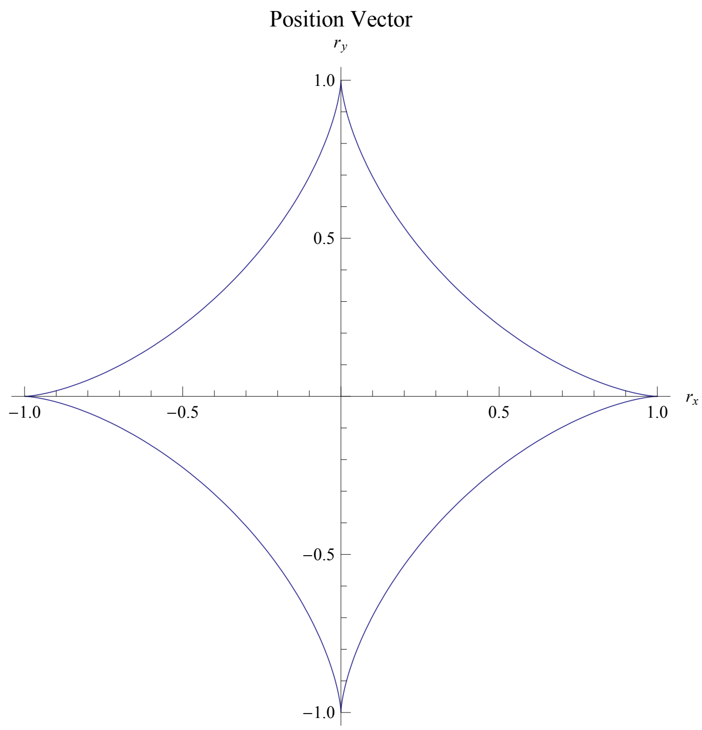
$$\frac{d}{d\,g}\left(-3\,c\,g^2\,h\right) = -6\,c\,g\,h;\ \frac{d}{d\,h}\left(3\,c\,h^2\,g\right) = 6\,c\,h\,g$$

and,

$$\vec{a} = \left(\frac{d}{d\,g}\left(-3\,c\,g^2\,h\right)\frac{d\,g}{d\,t},\ \frac{d}{d\,h}\left(3\,c\,h^2\,g\right)\frac{d\,h}{d\,t}\right) = \left(-6\,c\,g\,h\,\frac{d\,g}{d\,t},\ 6\,c\,h\,g\,\frac{d\,h}{d\,t}\right) =$$

$$(6\,c\,g\,h\sin t,\ 6\,c\,h\,g\cos t) = (6\,c\cos t\sin t\sin t,\ 6\,c\sin t\cos t\cos t) = \left(6\,c\cos t\sin^2 t\,,\ 6\,c\sin t\cos^2 t\right)$$

Out[43]=



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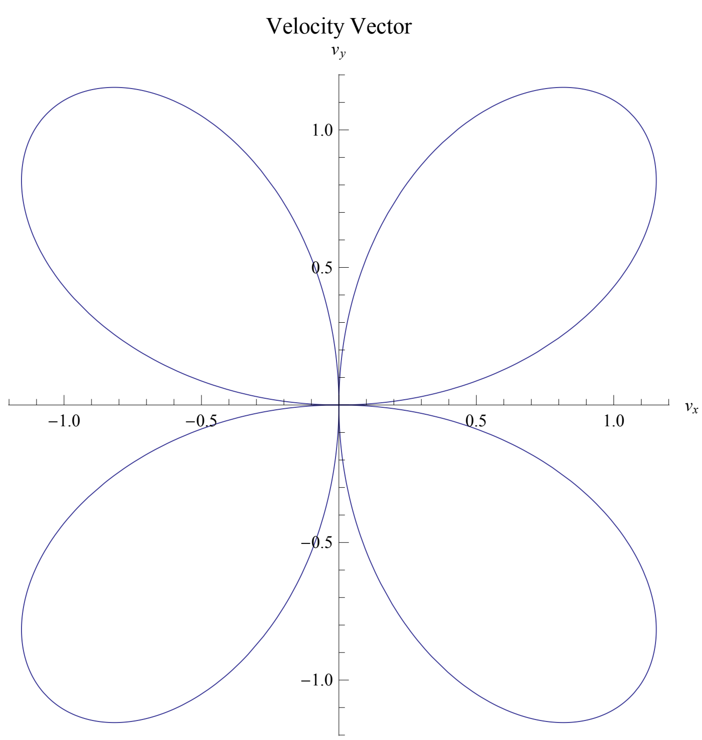

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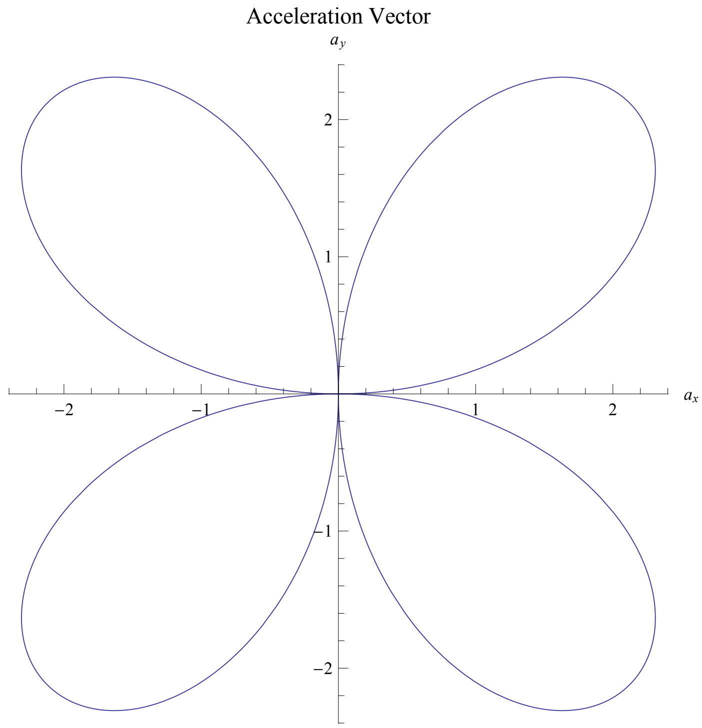
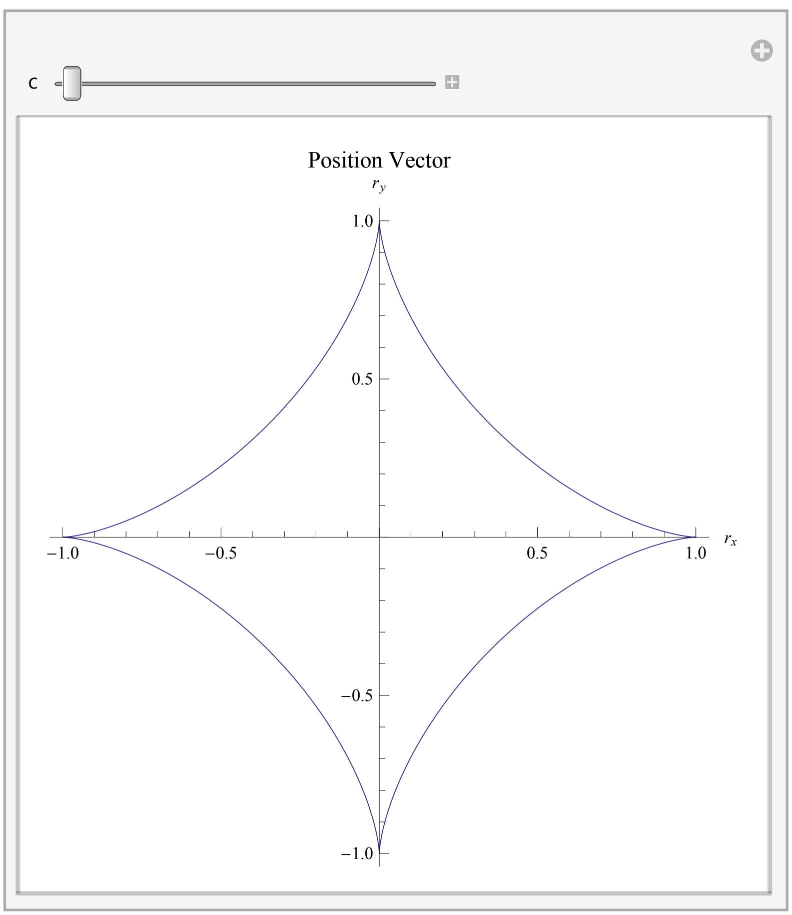


If you are viewing this with the CDF Player, you can manipulate the value of ω by moving the slider—watch the vertical axis scale change as you increase c,

Out[44]=



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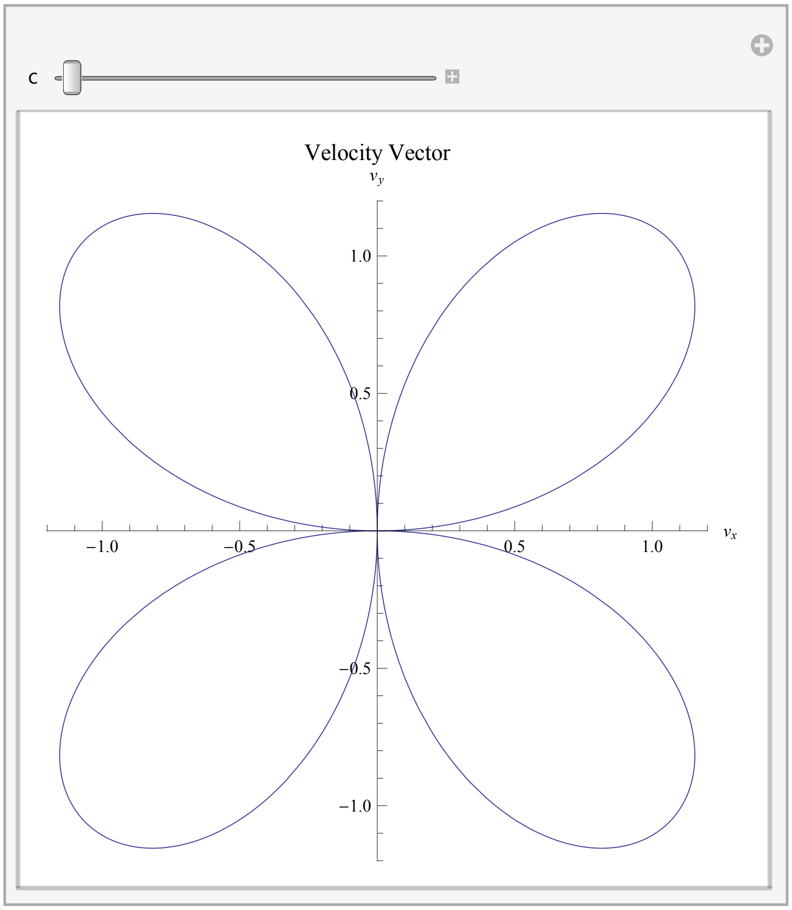

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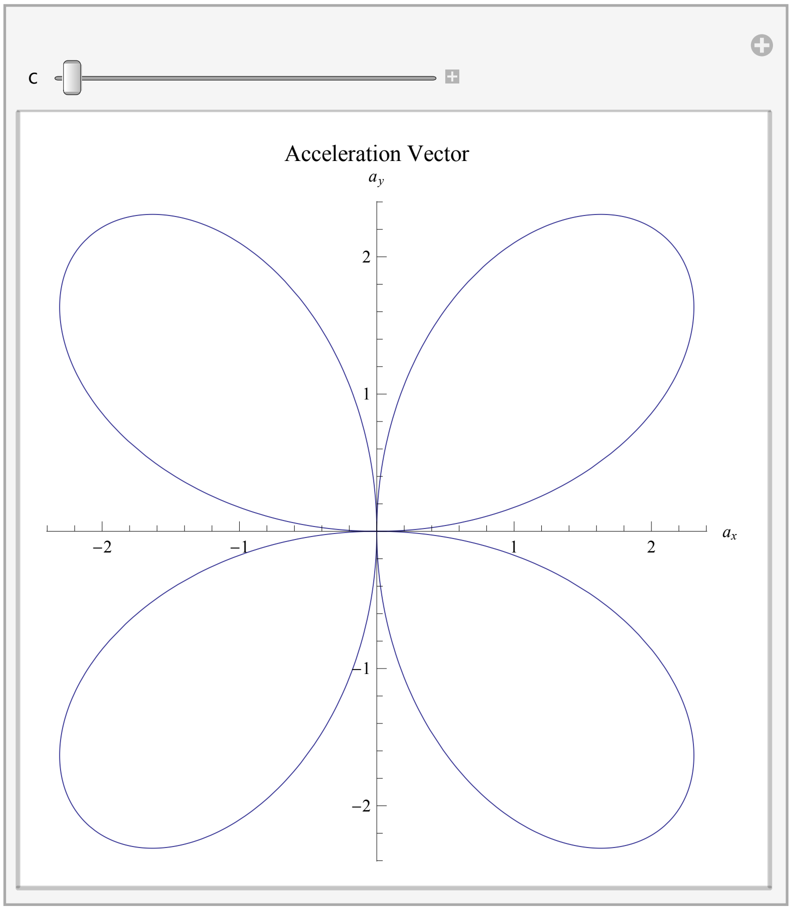


- $\vec{r} = (c\,(t - \sin t),\ c\,(1 - \cos\ t))$

The first task is to calculate the derivative of each component with respect to time, thus giving us the velocity vector:

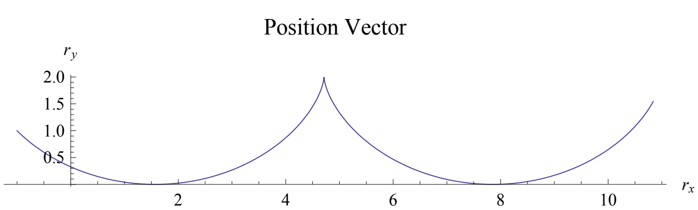
$$\vec{v} = \frac{d\,\vec{r}}{d\,t} = \frac{d}{d\,t}[c\,(t - \sin t),\ c\,(1 - \cos\ t)] = \left(\frac{d}{d\,t}\,c\,(t - \sin t),\ \frac{d}{d\,t}\,c\,(1 - \cos\ t)\right) =$$

$$\left(\frac{d}{d\,t}\,c\,t - \frac{d}{d\,t}\,c \sin t,\ \frac{d}{d\,t}\,c\ - \frac{d}{d\,t}\,c \cos\ t\right) = \left(c\,\frac{d}{d\,t}\,t - c\,\frac{d}{d\,t}\sin t,\ 0\ - c\,\frac{d}{d\,t}\cos\ t\right) = (c - c \cos t,\ c \sin t).$$

We then calculate the acceleration vector in a similar way,

$$\vec{a} = \frac{d\,\vec{v}}{d\,t} = \frac{d}{d\,t}\,(c - c \cos t,\ c \sin t) = \left(\frac{d}{d\,t}\,c - \frac{d}{d\,t}\,c \cos t,\ \frac{d}{d\,t}\,c \sin t\right) = \left(\frac{d}{d\,t}\,c - c\,\frac{d}{d\,t}\cos t,\ c\,\frac{d}{d\,t}\sin t\right) = (c \sin t,\ c \cos t).$$

Out[51]=



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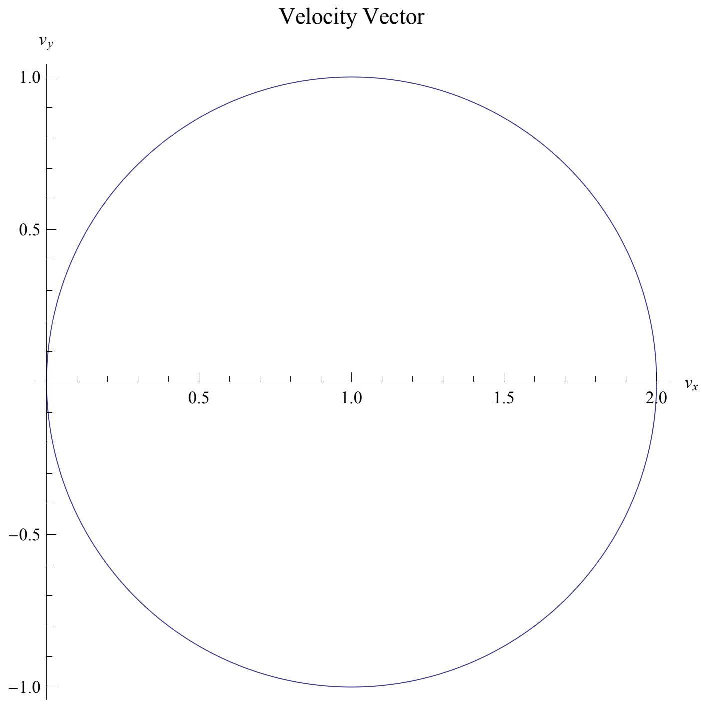

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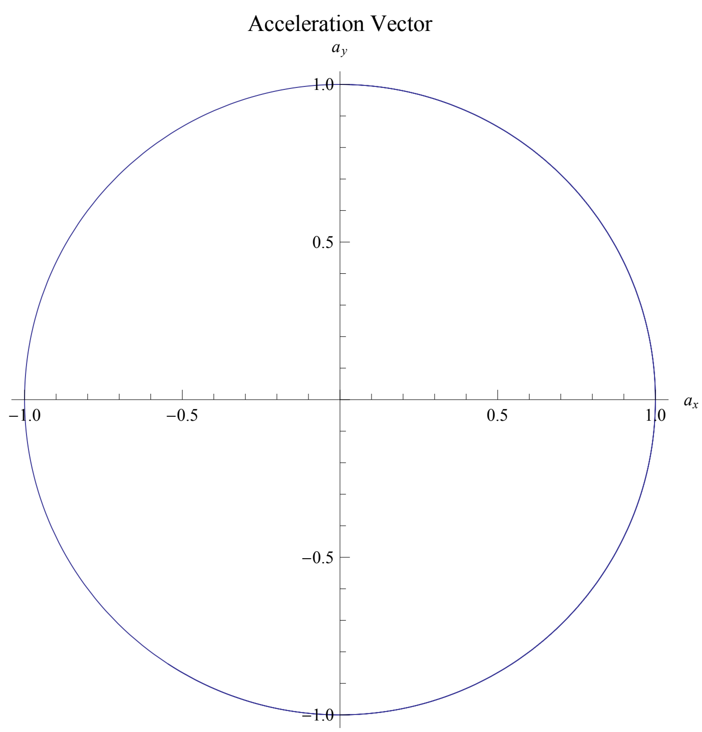
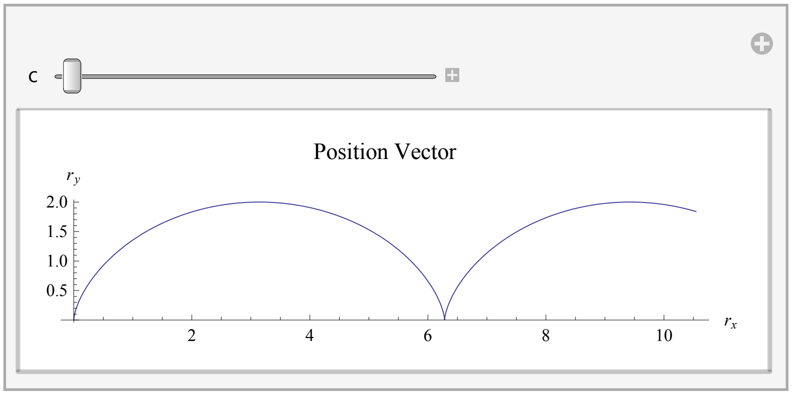


If you are viewing this with the CDF Player, you can manipulate the value of ω by moving the slider—watch the vertical axis scale change as you increase c,

Out[52]=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1.0
0.5
2
4
6
8
10
r_x

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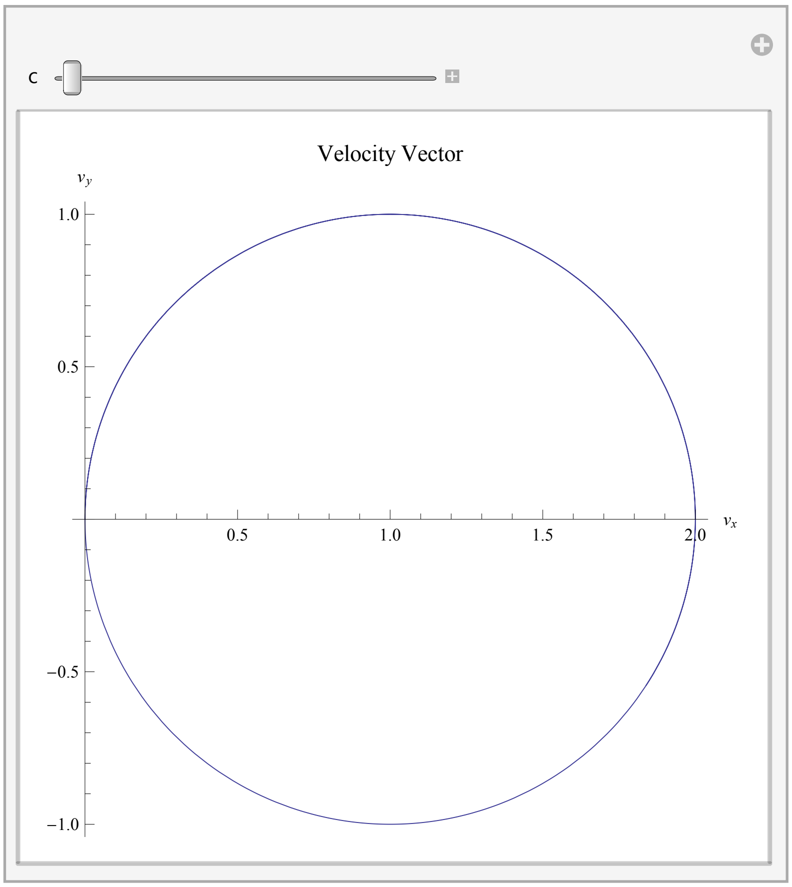
c
Velocity Vector
v_y
1.0
0.5
−0.5
−1.0
0.5
1.0
1.5
2.0
v_x

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