Lecture 2: Exercise 5

Explanation

Here we learn to apply the rules of differentiation to prove the validity of the differentiation formulas in Eq. (2).

Hint

To prove the two trigonometric derivatives we need to explore the nature of trigonometric functions and their limits. We will need to understand that sometimes a proof requires you to examine different cases and show that they are all the same.

Proving the exponential rule requires that you invent a new kind of differentiation, logarithmic differentiation. We will also need the chain rule.

Proving the logarithmic rule also requires logarithmic differentiation.

Answer

- $d/dt\,(\sin t) = \cos t$

We begin by noting the definition of the derivative,

$$\frac{d f}{d t} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

We know that $f(t + \Delta t) - f(t)$ is called Δf, so we write,

$$\begin{aligned}\Delta f &= f(t + \Delta t) - f(t)\\ &= \sin(t + \Delta t) - \sin(t).\end{aligned}$$

We need to figure out how to expand $\sin(t + \Delta t)$, if there is a way. Another way of asking this is, what is the sine of a sum of angles? If we look in a table of trigonometric formulas, the sine of a sum is,

$$\sin(t + \Delta t) = \sin t \cos \Delta t + \cos t \sin \Delta t.$$

We should always try to have a firm visualization of what we want to do in our minds, that is why it is helpful to draw a diagram. Sometimes we can't easily draw a picture or think our way through a problem. We need to invent arguments based—not on physical intuition—but on what the math is trying to tell us; this is one reason why theoretical physics is so hard.

The idea is to look at the problem and try to figure out what we need from experience and training. At first, this is very difficult and we often get things wrong—this is fine, we only really learn by getting it wrong.

You might be tempted to multiply the result to get,"

$$\sin(t + \Delta t) = \sin t + \sin \Delta t.$$

That would work if $\sin a$ was $\sin \times a$ and not the sine function applied to a. Here we have the sine function applied to the total $t + \Delta t$. That is not the same as $\sin t + \sin \Delta t$, as given by $\sin(t + \Delta t) = \sin t \cos \Delta t + \cos t \sin \Delta t$.

If you were to ask, "Is that because it is a function and not a variable?" The answer is yes.

Then we substitute this result into our formula for Δf:

$$\Delta f = \sin t \cos \Delta t + \cos t \sin \Delta t - \sin t.$$

We can reorder this,

$$\Delta f = \sin t \cos \Delta t - \sin t + \cos t \sin \Delta t.$$

We can install parentheses:

$$\Delta f = (\sin t \cos \Delta t - \sin t) + \cos t \sin \Delta t.$$

Then we can factor the quantity in parentheses,:

$$\Delta f = \sin t (\cos \Delta t - 1) + \cos t \sin \Delta t.$$

Following with the definition of the derivative:

$$\frac{\Delta f}{\Delta t} = \frac{\sin t (\cos \Delta t - 1) + \cos t \sin \Delta t}{\Delta t}.$$

We can rewrite this by separating the numerators since they have the same denominator,

$$\frac{\Delta f}{\Delta t} = \frac{\sin t (\cos \Delta t - 1)}{\Delta t} + \frac{\cos t \sin \Delta t}{\Delta t}$$

or, since a factor in the numerator is the same as a factor multiplying a fraction, we can rewrite this:

$$\frac{\Delta f}{\Delta t} = \sin t \frac{(\cos \Delta t - 1)}{\Delta t} + \cos t \frac{\sin \Delta t}{\Delta t}.$$

Now we can take the limits. To do this we first note that the limit of a sum is the sum of the limits:

$$\begin{aligned}\lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} &= \lim_{\Delta t \to 0} \left[\sin t \frac{(\cos \Delta t - 1)}{\Delta t} + \cos t \frac{\sin \Delta t}{\Delta t}\right]\\ &= \lim_{\Delta t \to 0} \left[\sin t \frac{(\cos \Delta t - 1)}{\Delta t}\right] + \lim_{\Delta t \to 0} \left(\cos t \frac{\sin \Delta t}{\Delta t}\right).\end{aligned}$$

Another rule of limits is that $\lim a x = a \lim x$, when you have a constant term a. In this case, since the limit is for Δt we can hold both $\sin t$ and $\cos t$ as fixed constants. So,

$$\lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} = \sin t \lim_{\Delta t \to 0} \left[\frac{(\cos \Delta t - 1)}{\Delta t}\right] + \cos t \lim_{\Delta t \to 0} \left(\frac{\sin \Delta t}{\Delta t}\right).$$

Now we run into a problem. $\frac{1}{\Delta t}$ where $\Delta t = 0$ is a major difficulty when you try to calculate something. Looking at this we can arbitrarily decide that the right-most term looks simplest.

Let's draw a figure to help us out. We can think of Δt as the angle swept out by rotating a segment from an origin point, O."

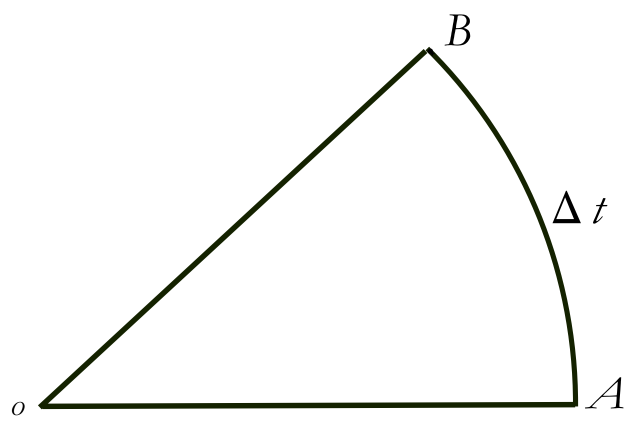
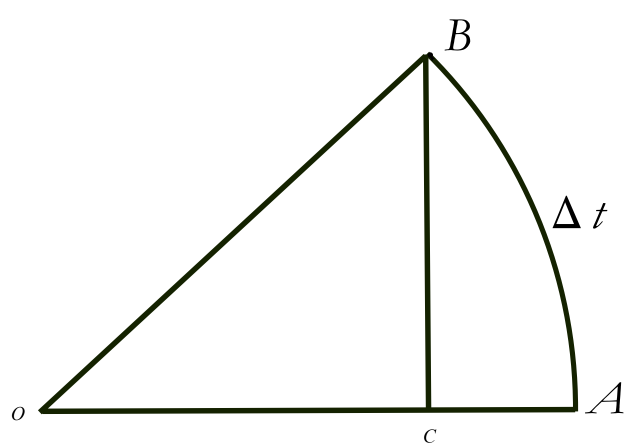


Figure 1: Diagram for Δt.

We also note that since,

$$\frac{\sin \Delta t}{\Delta t} = \frac{\sin(-\Delta t)}{-\Delta t}$$

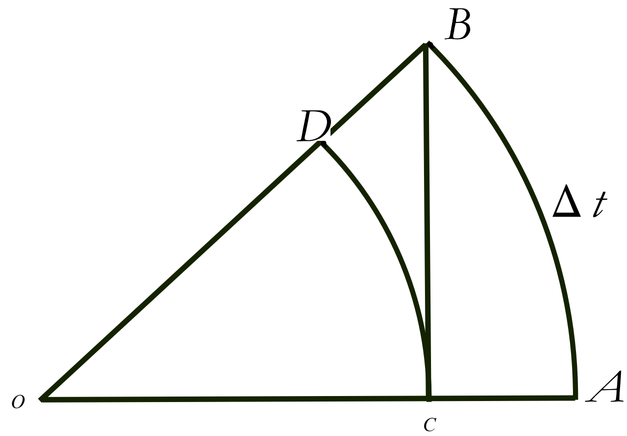
we need only worry about Δt being positive, the results will be the same for negative Δt. If we drop a perpendicular from B to segment OA we can label that intersection C."



We also note two facts from basic trigonometry,

$$\begin{aligned} OC &= \cos \Delta t \\ CB &= \sin \Delta t. \end{aligned}$$

The segment OC can also sweep out an arc, that arc intersects OB at the point D.



This gives us three regions. The sector of a circle AOB, the sector of the circle COD, and the triangle COB. We can relate the areas of these regions,

$$\text{Area of the sector } COD \le \text{Area of } \triangle COB \le \text{Area of the sector } AOB.$$

Since the area of a circular sector of length r and angle θ is $\frac{1}{2} r^2 \theta$, and the area of a right triangle with base b and height h is $\frac{1}{2} b\, h$ we note that,

$$A_{COD} = \frac{1}{2} \Delta t \cos^2 \Delta t$$

$$A_{COB} = \frac{1}{2} \cos \Delta t \sin \Delta t$$

$$A_{AOB} = \frac{1}{2} \Delta t.$$

We can then rewrite our inequality:

$$\frac{1}{2} \Delta t \cos^2 \Delta t \le \frac{1}{2} \cos \Delta t \sin \Delta t \le \frac{1}{2} \Delta t.$$

If we look at this a while to see how to simplify it, we come to the conclusion that we should divide by $\frac{1}{2} \Delta t \cos \Delta t$:

$$\cos \Delta t \le \frac{\sin \Delta t}{\Delta t} \le \frac{1}{\cos \Delta t}.$$

This is encouraging because we have the limit we are looking for in the center, $\frac{\sin \Delta t}{\Delta t}$. We know that as Δt approaches 0 (or $\Delta t \to 0$) that $\cos \Delta t \to 1$, since $\cos 0 = 1$. So,

$$1 \le \frac{\sin \Delta t}{\Delta t} \le \frac{1}{\cos \Delta t}.$$

Likewise we see that is also true for the right-hand side of our inequality, we can write that as $\Delta t \to 0$, $\frac{1}{\cos \Delta t} \to 1$. So,

$$1 \le \frac{\sin \Delta t}{\Delta t} \le 1.$$

Thus,

$$\lim_{\Delta t \to 0} \left(\frac{\sin \Delta t}{\Delta t} \right) = 1. \tag{0.1}$$

We proved this by showing that the limit was less than or equal to one, and that it was greater than or equal to one; thus it could only be equal to one. Now we need to prove the part that looked more complicated, the $\frac{\cos \Delta t - 1}{\Delta t}$ term. The first thing to recall, and this is very important; we can always perform any operations that leaves the entire form of our expression unchanged (multiplying by $\frac{a}{a}$, for example). All we need to do is choose the right *auxiliary expression*. In our case we multiply by the equivalent of $\frac{a}{a}$,

$$\frac{\cos \Delta t - 1}{\Delta t} = \frac{\cos \Delta t - 1}{\Delta t} \cdot \frac{\cos \Delta t + 1}{\cos \Delta t + 1}.$$

We expand this out algebraically to get,

$$\frac{\cos \Delta t - 1}{\Delta t} = \frac{\cos^2 \Delta t - 1}{\Delta t (\cos \Delta t + 1)}.$$

We know from section that $\sin^2 \theta + \cos^2 \theta = 1$, we can rewrite this $\sin^2 \theta = 1 - \cos^2 \theta$. We can reverse the sign of this expression to get $-\sin^2 \theta = \cos^2 \theta - 1$, which is what we want, so we can now write,

$$\frac{\cos \Delta t - 1}{\Delta t} = -\frac{\sin^2 \Delta t}{\Delta t (\cos \Delta t + 1)}.$$

Since $\sin^2 \theta = \sin \theta \sin \theta$ we can write,

$$\frac{\cos \Delta t - 1}{\Delta t} = -\frac{\sin \Delta t}{\Delta t} \cdot \frac{\sin \Delta t}{\cos \Delta t + 1}.$$

So, we are nearly home. Now we have to find the limit of this,

$$\lim_{\Delta t \to 0} \frac{\cos \Delta t - 1}{\Delta t} = \lim_{\Delta t \to 0} \left(-\frac{\sin \Delta t}{\Delta t} \cdot \frac{\sin \Delta t}{\cos \Delta t + 1} \right).$$

There is a theorem for limits that says that the limit of a product is the product of the limits, so we can rewrite this,

$$\lim_{\Delta t \to 0} \frac{\cos \Delta t - 1}{\Delta t} = \lim_{\Delta t \to 0} \left(-\frac{\sin \Delta t}{\Delta t} \right) \cdot \lim_{\Delta t \to 0} \frac{\sin \Delta t}{\cos \Delta t + 1}$$

or,

$$\lim_{\Delta t \to 0} \frac{\cos \Delta t - 1}{\Delta t} = -\lim_{\Delta t \to 0} \left(\frac{\sin \Delta t}{\Delta t} \right) \cdot \lim_{\Delta t \to 0} \frac{\sin \Delta t}{\cos \Delta t + 1}$$

We know that the limit of $\frac{\sin \Delta t}{\Delta t}$ is 1 as $\Delta t \to 0$, so,

$$\lim_{\Delta t \to 0} \frac{\cos \Delta t - 1}{\Delta t} = -1 \cdot \lim_{\Delta t \to 0} \frac{\sin \Delta t}{\cos \Delta t + 1}.$$

We now evaluate the limit on the right-hand side,

$$\lim_{\Delta t \to 0} \frac{\cos \Delta t - 1}{\Delta t} = -1 \cdot \frac{\sin 0}{\cos 0 + 1} = -1 \frac{0}{1 + 1} = 0.$$

Putting all of this together, we get,

$$\lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} = \sin t \lim_{\Delta t \to 0} \left[\frac{(\cos \Delta t - 1)}{\Delta t} \right] + \cos t \lim_{\Delta t \to 0} \left(\frac{\sin \Delta t}{\Delta t} \right),$$

$$= \sin t\,(0) + \cos t\,(1) = \cos t.$$

Thus we have proven that the derivative of $\sin t$ is $\cos t$.

You might be tempted to say something like, “I’m glad that was easy for you. You pulled of lot of theorems out of your—.”

Sure, I understand that. These things can get very involved. That is why it can be useful to go through them, they can force us to cover things we might not otherwise see.

■ $d/dt(\cos t) = -\sin t$

Again, we begin with,

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\cos(t + \Delta t) - \cos(t)}{\Delta t}.$$

Then we write,

$$\cos(t + \Delta t) = \cos t \cos \Delta t - \sin t \sin \Delta t.$$

Thus,

$$\begin{aligned} f(t + \Delta t) - f(t) &= \cos t \cos \Delta t - \sin t \sin \Delta t - \cos t \\ &= \cos t \cos \Delta t - \cos t - \sin t \sin \Delta t \\ &= (\cos t \cos \Delta t - \cos t) - \sin t \sin \Delta t \\ &= \cos t (\cos \Delta t - 1) - \sin t \sin \Delta t \end{aligned}$$

Then,

$$\begin{aligned} \frac{f(t + \Delta t) - f(t)}{\Delta t} &= \frac{\cos t (\cos \Delta t - 1) - \sin t \sin \Delta t}{\Delta t} \\ &= \frac{\cos t (\cos \Delta t - 1)}{\Delta t} - \frac{\sin t \sin \Delta t}{\Delta t} \\ &= \cos t \frac{(\cos \Delta t - 1)}{\Delta t} - \sin t \frac{\sin \Delta t}{\Delta t} \end{aligned}$$

Leaving us with,

$$\lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} = \cos t \lim_{\Delta t \to 0} \frac{(\cos \Delta t - 1)}{\Delta t} - \sin t \lim_{\Delta t \to 0} \frac{\sin \Delta t}{\Delta t}$$

We can use the results from the last section to get,

$$\lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} = \cos t (0) - \sin t (1) = -\sin t.$$

▪ $d/dt\left(e^t\right) = e^t$

We begin by writing

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{e^{(t+\Delta t)} - e^t}{\Delta t}$$

then,

$$f(t + \Delta t) - f(t) = e^{(t+\Delta t)} - e^t$$

by the fact that $a^{b+c} = a^b a^c$,

$$f(t + \Delta t) - f(t) = e^t e^{\Delta t} - e^t.$$

We can factor e^t out of this,

$$f(t + \Delta t) - f(t) = e^t \left(e^{\Delta t} - 1\right).$$

Now we have,

$$\frac{f(t + \Delta t) - f(t)}{\Delta t} = \frac{e^t (e^{\Delta t} - 1)}{\Delta t}.$$

We pull out the e^t,

$$\frac{f(t + \Delta t) - f(t)}{\Delta t} = e^t \frac{(e^{\Delta t} - 1)}{\Delta t}$$

We expand this,

$$\frac{f(t + \Delta t) - f(t)}{\Delta t} = e^t \left(\frac{e^{\Delta t}}{\Delta t} - \frac{1}{\Delta t}\right).$$

Then,

$$\lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} = e^t \lim_{\Delta t \to 0} \left(\frac{e^{\Delta t}}{\Delta t} - \frac{1}{\Delta t} \right)$$

$$= e^t \lim_{\Delta t \to 0} \frac{e^{\Delta t}}{\Delta t} - e^t \lim_{\Delta t \to 0} \frac{1}{\Delta t}$$

It looks like we are dividing by 0.

$$\lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} = e^t \left(\frac{1}{0} - \frac{1}{0} \right)$$

Where $\left(\frac{1}{0} - \frac{1}{0}\right)$ is a meaningless result.

We have just discovered that we need a new kind of differentiation. It is called *logarithmic differentiation*. It relies on the fact that exponentiation and logarithms are inverse operations. We can change our original expression by taking its logarithm,

$$\ln f = \ln e^t.$$

Since $\ln a^x = x \ln a$, we can write,

$$\ln f = t \ln e.$$

Since $\ln e = 1$, we can write,

$$\ln f = t.$$

We need to find the derivative of $\ln f$ and set it equal to the derivative t with respect to t. This is, in fact, then next problem, kind of sneaky—but that is the way of mathematics, sometimes you need the answer to another problem before you can proceed.

So the derivative of $\ln f$ with respect to t is,

$$\frac{d(\ln f)}{dt} = \lim_{\Delta t \to 0} \frac{\ln(t + \Delta t) - \ln t}{\Delta t},$$

and,

$$\frac{f(t + \Delta t) - f(t)}{\Delta t} = \frac{\ln(t + \Delta t) - \ln t}{\Delta t}$$

Since, $\ln x - \ln t = \ln \frac{x}{t}$, we can write

$$\frac{\ln(t + \Delta t) - \ln(t)}{\Delta t} = \frac{1}{\Delta t} \ln \frac{(t + \Delta t)}{t}$$

$$= \frac{1}{\Delta t} \ln \left(\frac{t}{t} + \frac{\Delta t}{t} \right)$$

$$= \frac{1}{\Delta t} \ln \left(1 + \frac{\Delta t}{t} \right).$$

You can look up the definition of e in a calculus book, and it is almost what we have here,

$$\lim_{\Delta t \to 0} \left(1 + \frac{\Delta \mathrm{t}}{t} \right)^{\frac{t}{\Delta t}} = e.$$

To get from this to what we have we can work backwards, noting that $a \ln x = \ln x^a$,

$$\ln \left(1 + \frac{\Delta \mathrm{t}}{t} \right)^{\frac{t}{\Delta t}} = \frac{t}{\Delta t} \ln \left(1 + \frac{\Delta t}{t} \right).$$

So we have to change out original factor of $\frac{1}{\Delta t}$ into $\frac{t}{\Delta t}$. To do this we multiply by the auxiliary expression t/t, so,

$$\frac{\ln(t + \Delta t) - \ln(t)}{\Delta t} = \frac{t}{t} \frac{1}{\Delta t} \ln \left(1 + \frac{\Delta t}{t} \right)$$

$$= \frac{1}{t} \frac{t}{\Delta t} \ln \left(1 + \frac{\Delta t}{t} \right)$$

We know then that,

$$\frac{\ln(t + \Delta t) - \ln(t)}{\Delta t} = \frac{1}{t} \ln\left(1 + \frac{\Delta t}{t}\right)^{\frac{t}{\Delta t}}.$$

We then apply the definition,

$$\lim_{\Delta t \to 0} \left(1 + \frac{\Delta t}{t}\right)^{\frac{t}{\Delta t}} = e,$$

To get,

$$\lim_{\Delta t \to 0} \frac{\ln(t + \Delta t) - \ln(t)}{\Delta t} = \frac{1}{t} \ln \lim_{\Delta t \to 0} \left(1 + \frac{\Delta t}{t}\right)^{\frac{t}{\Delta t}}.$$

$$= \frac{1}{t} \ln e = \frac{1}{t}.$$

So we have proven that $\frac{d}{dt}(\ln t) = \frac{1}{t}$. Now we apply this to the derivative of the logarithm of our function,

$$\frac{d \ln f}{d t} = \frac{1}{f}$$

since f is an arbitrary function label we cannot gaurantee this it does not have its own derivative. We think about that for a minute and we generalize what we have written,

$$\frac{d \ln f}{d t} = \frac{d \text{ function of } f}{d t}.$$

We can lable this function anything we want, so let's call it $g(f)$.

$$\frac{d \ln f}{d t} = \frac{d g(f)}{d t}.$$

The problem is that we now have an expression that is not specifically dependent on t. We can rescue the situation by multiplying by $\frac{d f}{d f}$,

$$\frac{d g(f)}{d t} = \frac{d g(f)}{d t} \frac{d f}{d f}.$$

Now we can reorder the terms,

$$\frac{d g(f)}{d t} = \frac{d g(f)}{d f} \frac{d f}{d t}.$$

In general whenever we have the derivative of a function of an arbitrary function, we always multiply the derivative of the function by the derivative of the arbitrary function. This is the *chain rule*. Since we know that $\frac{d g(f)}{d f} = \frac{1}{f}$, so we now have,

$$\frac{d \ln f}{d t} = \frac{1}{f} \frac{d f}{d t}.$$

So, recalling that we needed to set the derivative of $\ln f$ equal to one, we write,

$$\frac{1}{f} \frac{d f}{d t} = 1$$

or,

$$\frac{d f}{d t} = f,$$

since $f = e^t$, we have,

$$\frac{d e^t}{d t} = e^t.$$

In one exercise we have derived the chain rule, and proved both $\frac{d}{dt}(\ln t) = \frac{1}{t}$ and $\frac{d}{dt} e^t = e^t$."

- $\boldsymbol{d/dt\,(\ln t) = \frac{1}{t}}$

We proved this as an itnermediate result in the previous section.