Lecture 2: Exercise 1

Explanation

This is an exercise in applying the process of differentiation to several functions. It is a straightforward calculation.

Hint

Apply the definition of differentiation step-by-step.

Answer

- $\boldsymbol{f(t) = t^4 + 3\,t^3 - 12\,t^2 + t\, - 6.}$

We first define

$$f(t + \Delta\, t) = (t + \Delta\, t)^4 + 3\,(t + \Delta\, t)^3 - 12\,(t + \Delta\, t)^2 + (t + \Delta\, t)\, - 6$$

$$\begin{aligned} f(t + \Delta\, t) = {} & \Delta\, t^4 + t^4 + 4\,\Delta\, t\, t^3 + 6\,\Delta\, t^2\, t^2 + 4\,\Delta\, t^3\, t + 3\,\Delta\, t^3 + 3\,t^3 \\ & + 9\,\Delta\, t\, t^2 + 9\,\Delta\, t^2\, t - 12\,\Delta\, t^2 - 12\,t^2 - 24\,\Delta\, t\, t + t + \Delta\, t\, - 6 \end{aligned}$$

then we write,

$$f(t+\Delta t)-f(t)=\Delta t^4+t^4+4\,\Delta t\,t^3+6\,\Delta t^2\,t^2+4\,\Delta t^3\,t+3\,\Delta t^3+3\,t^3$$
$$+9\,\Delta t\,t^2+9\,\Delta t^2\,t-12\,\Delta t^2-12\,t^2-24\,\Delta t\,t+t+\Delta t$$
$$-6-t^4-3\,t^3+12\,t^2-t+6$$

$$f(t+\Delta t)-f(t)=\Delta t^4+4\,\Delta t\,t^3+6\,\Delta t^2\,t^2+4\,\Delta t^3\,t+3\,\Delta t^3$$
$$+9\,\Delta t\,t^2+9\,\Delta t^2\,t-12\,\Delta t^2-24\,\Delta t\,t+\Delta t$$

then we divide by Δt,

$$\frac{f(t+\Delta t)-f(t)}{\Delta t}=\Delta t^3+4\,t^3+6\,\Delta t\,t^2+4\,\Delta t^2\,t+3\,\Delta t^2$$
$$+9\,t^2+9\,\Delta t\,t-12\,\Delta t-24\,t+1$$

then we take the limit as $\Delta t\to 0$,

$$\lim_{\Delta t\to 0}\frac{f(t+\Delta t)-f(t)}{\Delta t}=4\,t^3+9\,t^2-24\,t+1.$$

this is the derivative.

- **$g(x)=\sin x-\cos x$.**

Here we using the sum rule and the results of Eq. (2),

$$g'(x)=\cos x+\sin x.$$

- **$\theta(\alpha)=e^\alpha+\alpha\ln\alpha$.**

We begin by using the sum rule

$$\theta'(\alpha)=\frac{d}{d\,\alpha}e^\alpha+\frac{d}{d\,\alpha}(\alpha\ln\alpha).$$

We then apply Eq. (2) for the first term,

$$\theta'(\alpha)=e^\alpha+\frac{d}{d\,\alpha}(\alpha\ln\alpha).$$

We then apply the product rule for the second term,

$$\theta'(\alpha)=e^\alpha+\alpha\frac{d}{d\,\alpha}\ln\alpha+\ln\alpha\frac{d}{d\,\alpha}\alpha\,.$$

We then apply Eq. 2 for the second term,

$$\theta'(\alpha)=e^\alpha+\frac{\alpha}{\alpha}+\ln\alpha\frac{d}{d\,\alpha}\alpha=e^\alpha+1+\ln\alpha\frac{d}{d\,\alpha}\alpha$$

Since $d\,\alpha/d\,\alpha=1$,

$$\theta'(\alpha)=e^\alpha+1+\ln\alpha.$$

- **$x(t)=\sin^2 t-\cos t$.**

Here we apply the sum rule,

$$x'(t)=\frac{d}{d\,t}\sin^2 t-\frac{d}{d\,t}\cos t.$$

We complete the second term, as it is simpler,

$$x'(t)=\frac{d}{d\,t}\sin^2 t-(-\sin t)=\frac{d}{d\,t}\sin^2 t+\sin t.$$

The second term is a little tricky. The easy way is to apply the chain rule. We define a new variable, $u=\sin t$, then

$$\frac{d}{d\,t}\sin^2 t=\frac{d}{d\,u}u^2\,\frac{d}{d\,t}(\sin t)=2\,u\cos t$$

we then substitute the value of u,

$$\frac{d}{dt}\sin^2 t = 2\sin t\cos t.$$

So,

$$x'(t) = 2\sin t\cos t + \sin t.$$