Interlude 3: Exercise 2

Explanation

A useful skill, for a theoretical physicists, is to examine specific points in space, or in data, to determine their qualities. Here we test three points against the behavior of two functions to determine their properties.

Hint

You will need to know about partial derivatives and how to apply the Hessian determinant..

Answer

- $\boldsymbol{F(x, y) = \sin x + \sin y}$

In order to determine if a point is a stationary point with respect to a function, we must first find the partial derivatives of that function,

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\sin x + \frac{\partial}{\partial x}\sin y = \cos x$$

and

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\sin x + \frac{\partial}{\partial y}\sin y = \cos y.$$

We then insert our first point,

$$\frac{\partial F(\pi/2, -\pi/2)}{\partial x} = \cos\frac{\pi}{2} = 0$$

$$\frac{\partial F(\pi/2, -\pi/2)}{\partial y} = -\cos\frac{\pi}{2} = 0$$

This is a stationary point. What kind of stationary point? We now have to take the second derivatives,

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial}{\partial x}\cos x = -\sin x$$

for the point is question this is -1.

$$\frac{\partial^2 F}{\partial y^2} = \frac{\partial}{\partial y}\cos y = -\sin y$$

for the point in question this is 1

$$\frac{\partial^2 F}{\partial x\,\partial y} = \frac{\partial}{\partial y}\cos x = 0.$$

The Hessian determinant is then

$$\text{Det}\,H = \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial y^2} - \left(\frac{\partial^2 F}{\partial x\,\partial y}\right)^2 = \sin x \sin y$$

and the trace is

$$\text{Tr}\,H = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = -\sin x - \sin y.$$

For our point these become

$$\text{Det}\,H = \sin x \sin y = -1,\ \ \text{Tr}\,H = -\sin x - \sin y = 0.$$

Since the determinant is negative, we have a saddle point at $\left(x = \frac{\pi}{2},\ y = -\frac{\pi}{2}\right)$. We now examine the second point,

$$\frac{\partial F(-\pi/2, \pi/2)}{\partial x} = -\cos\frac{\pi}{2} = 0$$

$$\frac{\partial F(-\pi/2, \pi/2)}{\partial y} = \cos\frac{\pi}{2} = 0$$

$$\frac{\partial^2 F(-\pi/2, \pi/2)}{\partial x^2} = -1$$

$$\frac{\partial^2 F(-\pi/2, \pi/2)}{\partial y^2} = 1$$

$$\frac{\partial^2 F}{\partial x\,\partial y} = 0.$$

$$\text{Det}\,H = \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial y^2} - \left(\frac{\partial^2 F}{\partial x\,\partial y}\right)^2 = -1$$

This gives us another saddle point. The third point gives us

$$\frac{\partial F(-\pi/2, -\pi/2)}{\partial x} = -\cos\frac{\pi}{2} = 0$$

$$\frac{\partial F(-\pi/2,\ -\pi/2)}{\partial y} = -\cos\frac{\pi}{2} = 0$$

$$\frac{\partial^2 F(-\pi/2,\ \pi/2)}{\partial x^2} = 1$$

$$\frac{\partial^2 F(-\pi/2,\ \pi/2)}{\partial y^2} = 1$$

$$\frac{\partial^2 F}{\partial x\, \partial y} = 0.$$

$$\text{Det}\, H = \frac{\partial^2 F}{\partial x^2}\ \frac{\partial^2 F}{\partial y^2} - \left(\frac{\partial^2 F}{\partial x\, \partial y}\right)^2 = 1$$

$$\text{Tr}\, H = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = -\sin\left(\frac{-\pi}{2}\right) - \sin\left(\frac{-\pi}{2}\right) = -(-1) - (-1) = 2$$

So the third point is a local minimum.

- ***F*(*x*, *y*) = cos *x* + cos *y***

For the first point we have

$$\frac{\partial F(\pi/2,\ -\pi/2)}{\partial x} = -\sin\frac{\pi}{2} = -1$$

$$\frac{\partial F(-\pi/2,\ -\pi/2)}{\partial y} = -\sin\frac{\pi}{2} = 1$$

this is not a stationary point. In fact none of them are.