# Interlude 3: Exercise 1

## Explanation

This exercise provides practice in finding partial derivatives.

## Hint

You will need to know the power rule, the sum rule, the sine and cosine rules, the exponential rule, and the chain rule.

## Answer

- **$x^2 + y^2 = \sin(x\,y)$**

There are two ways of doing this problem, the first is to rerwrite it,

$$x^2 + y^2 - \sin(x\,y) = 0.$$

The second is to leave it as it is. In either case we will be differentiating both sides of the equation at the same time.

$$\frac{\partial}{\partial x}\left[x^2 + y^2 = \sin(x\,y)\right] = \left[\frac{\partial}{\partial x}x^2 + \frac{\partial}{\partial x}y^2 = \frac{\partial}{\partial x}\sin(x\,y)\right]$$
$$= [2\,x = y\cos(x\,y)]$$

Similarly,

$$\frac{\partial}{\partial y}\left[x^2 + y^2 = \sin(x\,y)\right] = \left[\frac{\partial}{\partial y}x^2 + \frac{\partial}{\partial y}y^2 = \frac{\partial}{\partial y}\sin(x\,y)\right]$$
$$= [2\,y = x\cos(x\,y)]$$

Then we have,

$$\frac{\partial^2}{\partial x^2}\left[x^2 + y^2 = \sin(x\,y)\right] = \left[\frac{\partial}{\partial x}2\,x = \frac{\partial}{\partial x}y\cos(x\,y)\right]$$
$$= \left[2 = y\,\frac{\partial}{\partial x}\cos(x\,y) + \cos(x\,y)\,\frac{\partial}{\partial x}y\right]$$
$$= \left[2 = -y^2\ \sin(x\,y)\right]$$

and

$$\frac{\partial^2}{\partial x\,\partial y}\left[x^2 + y^2 = \sin(x\,y)\right] = \left[\frac{\partial}{\partial y}2\,x = \frac{\partial}{\partial y}y\cos(x\,y)\right]$$
$$= \left[0 = y\,\frac{\partial}{\partial y}\cos(x\,y) + \cos(x\,y)\,\frac{\partial}{\partial y}y\right]$$
$$= [0 = -y\,x\sin(x\,y) + \cos(x\,y)\,]$$

We also have

$$\frac{\partial^2}{\partial y^2}\left[x^2 + y^2 = \sin(x\,y)\right] = \left[\frac{\partial}{\partial y}2\,y = \frac{\partial}{\partial y}x\cos(x\,y)\right] = \left[2 = -x^2\sin(x\,y)\right]$$

and

$$\frac{\partial^2}{\partial x\,\partial y}\left[x^2 + y^2 = \sin(x\,y)\right] = \left[\frac{\partial}{\partial x}2\,y = \frac{\partial}{\partial x}x\cos(x\,y)\right]$$
$$= \left[0 = x\,\frac{\partial}{\partial x}\cos(x\,y) + \cos(x\,y)\,\frac{\partial}{\partial x}x\right]$$
$$= [0 = -x\,y\sin(x\,y) + \cos(x\,y)]$$

Note that $\partial^2 / \partial x\,\partial y$ is independent of the order of differentiation.

- $\frac{x}{y}\,e^{x^2+y^2}$

We use the constant multiple rule here,

$$\frac{\partial}{\partial x}\frac{x}{y}e^{x^2+y^2} = \frac{1}{y}\frac{\partial}{\partial x}x\,e^{x^2+y^2}$$

We then apply the product rule,

$$\frac{\partial}{\partial x}\frac{x}{y}e^{x^2+y^2} = \frac{1}{y}\left(x\,\frac{\partial}{\partial x}e^{x^2+y^2} + e^{x^2+y^2}\,\frac{\partial}{\partial x}x\right)$$

We can simplify this by using the chain rule. We write

$$a = x^2 + y^2$$

so we have

$$\frac{\partial}{\partial x}\frac{x}{y}e^{x^2+y^2} = \frac{1}{y}\left(x\,\frac{\partial}{\partial a}e^a + e^a\,\frac{\partial}{\partial x}x\right)$$

or

$$\frac{\partial}{\partial x}\frac{x}{y}e^{x^2+y^2}=\frac{1}{y}(x\ e^a+e^a)$$

we also have

$$\frac{\partial}{\partial x}a=2x$$

so,

$$\frac{\partial}{\partial x}\frac{x}{y}e^{x^2+y^2}=\frac{1}{y}\left(2x^2\ e^{x^2+y^2}+e^{x^2+y^2}\right)=\frac{e^{x^2+y^2}}{y}\left(2x^2\ +1\right).$$

Similarly,

$$\frac{\partial}{\partial y}\frac{x}{y}e^{x^2+y^2}=\frac{x}{y}\frac{\partial}{\partial y}e^{x^2+y^2}+e^{x^2+y^2}\ \frac{\partial}{\partial y}\frac{x}{y}$$

$$\frac{\partial}{\partial y}\frac{x}{y}e^{x^2+y^2}=\frac{x}{y}\frac{\partial}{\partial y}e^a+e^a\ \frac{\partial}{\partial y}\frac{x}{y}$$

$$\frac{\partial}{\partial y}\frac{x}{y}e^{x^2+y^2}=\frac{x}{y}e^a-e^a\ \frac{x}{y^2}$$

where

$$\frac{\partial}{\partial y}a=2y$$

or

$$\frac{\partial}{\partial y}\frac{x}{y}e^{x^2+y^2}=2y\frac{x}{y}e^{x^2+y^2}-e^{x^2+y^2}\ \frac{x}{y^2}$$

so

$$\frac{\partial}{\partial y}\frac{x}{y}e^{x^2+y^2}=2x\,e^{x^2+y^2}-e^{x^2+y^2}\ \frac{x}{y^2}=x\,e^{x^2+y^2}\left(2\ -\frac{1}{y^2}\right).$$

Then

$$\frac{\partial^2}{\partial x^2}\frac{x}{y}e^{x^2+y^2}=\frac{1}{y}\frac{\partial}{\partial x}e^{x^2+y^2}\left(2x^2\ +1\right)$$

$$\frac{\partial^2}{\partial x^2}\frac{x}{y}e^{x^2+y^2}=\frac{2x^2\ +1}{y}\frac{\partial}{\partial x}e^{x^2+y^2}\ +\frac{e^{x^2+y^2}}{y}\frac{\partial}{\partial x}\left(2x^2\ +1\right)$$

$$\frac{\partial^2}{\partial x^2}\frac{x}{y}e^{x^2+y^2}=\frac{2x^2\ +1}{y}e^{x^2+y^2}\ \frac{\partial}{\partial x}\left(x^2+y^2\right)+\frac{e^{x^2+y^2}}{y}4x$$

$$\frac{\partial^2}{\partial x^2}\frac{x}{y}e^{x^2+y^2}=\frac{2x^2\ +1}{y}e^{x^2+y^2}\ 2x+\frac{e^{x^2+y^2}}{y}4x$$

$$\frac{\partial^2}{\partial x^2}\frac{x}{y}e^{x^2+y^2}=\frac{e^{x^2+y^2}}{y}\left(4x^3\ +6x\right).$$

Then,

$$\frac{\partial^2}{\partial x\,\partial y}\frac{x}{y}e^{x^2+y^2}=\frac{\partial}{\partial\,y}\frac{e^{x^2+y^2}}{y}\left(2x^2\ +1\right)$$

$$\frac{\partial^2}{\partial x\,\partial y}\frac{x}{y}e^{x^2+y^2}=\left(2x^2\ +1\right)\frac{\partial}{\partial\,y}\frac{e^{x^2+y^2}}{y}+\frac{e^{x^2+y^2}}{y}\frac{\partial}{\partial\,y}\left(2x^2\ +1\right)$$

$$\frac{\partial^2}{\partial x\,\partial y}\frac{x}{y}e^{x^2+y^2}=\left(2x^2\ +1\right)\left(e^{x^2+y^2}\ \frac{\partial}{\partial\,y}y^{-1}+y^{-1}\ \frac{\partial}{\partial\,y}e^{x^2+y^2}\right)$$

$$\frac{\partial^2}{\partial x\,\partial y}\frac{x}{y}e^{x^2+y^2} = \left(2x^2\ +1\right)\left[-e^{x^2+y^2}y^{-2} + y^{-1}e^{x^2+y^2}\frac{\partial}{\partial y}\left(x^2+y^2\right)\right]$$

$$\frac{\partial^2}{\partial x\,\partial y}\frac{x}{y}e^{x^2+y^2} = \left(2x^2\ +1\right)\left(2\,e^{x^2+y^2} - e^{x^2+y^2}y^{-2}\right)$$

$$\frac{\partial^2}{\partial x\,\partial y}\frac{x}{y}e^{x^2+y^2} = \left(2x^2\ +1\right)\left(2\,y^2\,y^{-2}\,e^{x^2+y^2} - e^{x^2+y^2}y^{-2}\right)$$

$$\frac{\partial^2}{\partial x\,\partial y}\frac{x}{y}e^{x^2+y^2} = \left(2x^2\ +1\right)\frac{e^{x^2+y^2}}{y^2}\left(2\,y^2-1\right)$$

We also have

$$\frac{\partial^2}{\partial y^2}\frac{x}{y}e^{x^2+y^2} = \frac{\partial}{\partial y}x\,e^{x^2+y^2}\left(2\ -\frac{1}{y^2}\right)$$

$$\frac{\partial^2}{\partial y^2}\frac{x}{y}e^{x^2+y^2} = x\,e^{x^2+y^2}\frac{\partial}{\partial y}\left(2\ -\frac{1}{y^2}\right) + \left(2\ -\frac{1}{y^2}\right)\frac{\partial}{\partial y}x\,e^{x^2+y^2}$$

$$\frac{\partial^2}{\partial y^2}\frac{x}{y}e^{x^2+y^2} = -\frac{2\,x\,e^{x^2+y^2}}{y^3} + \left(2\ -\frac{1}{y^2}\right)x\,e^{x^2+y^2}\frac{\partial}{\partial y}\left(x^2+y^2\right)$$

$$\frac{\partial^2}{\partial y^2}\frac{x}{y}e^{x^2+y^2} = -\frac{2\,x\,e^{x^2+y^2}}{y^3} + \left(2\ -\frac{1}{y^2}\right)2\,y\,x\,e^{x^2+y^2}$$

$$\frac{\partial^2}{\partial y^2}\frac{x}{y}e^{x^2+y^2} = -\frac{2\,x\,e^{x^2+y^2}}{y^3} + 4\,y\,x\,e^{x^2+y^2} - \frac{2\,y\,x\,e^{x^2+y^2}}{y^2}$$

$$\frac{\partial^2}{\partial y^2}\frac{x}{y}e^{x^2+y^2} = 2\,x\,e^{x^2+y^2}\left(2\,y\ -\frac{1}{y}-\frac{1}{y^3}\right)$$

$$\frac{\partial^2}{\partial y^2}\frac{x}{y}e^{x^2+y^2} = \frac{2\,x\,e^{x^2+y^2}}{y^3}\left(1-y^2+2\,y^4\right).$$

### ■ $e^x \cos y$

Here we will employ the constant rule,

$$\frac{\partial}{\partial x}e^x\cos y = \cos y\,\frac{\partial}{\partial x}e^x = e^x\cos y$$

and

$$\frac{\partial}{\partial y}e^x\cos y = e^x\,\frac{\partial}{\partial y}\cos y = -e^x\sin y$$

then

$$\frac{\partial^2}{\partial x^2}e^x\cos y = \frac{\partial}{\partial x}e^x\cos y\ = e^x\cos y$$

and

$$\frac{\partial^2}{\partial y^2}e^x\cos y = -e^x\,\frac{\partial}{\partial y}\sin y = -e^x\cos y$$

and finally,

$$\frac{\partial^2}{\partial x\,\partial y}e^x\cos y = \frac{\partial}{\partial y}e^x\cos y\ = -e^x\sin y.$$