Logic and Proof

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Introduction

In what follows, I will begin by giving an overview of logic. Then I will present a large number of arguments by logic that can be used in proofs, I will also give an example of a proof by truth tables and a more traditional proof. Then I will list my conclusions and references.

Logic and proof

To begin with, I will need to present the basic method of formal mathematics. I will try to make this as basic as possible and still be useful. It is important to realize that in mathematics, until an idea is applied to something concrete ite no meaning. Thus mathematics is the ultimate abstraction from reality; we speak of pure ideas without regard to meaning. It is best to think of mathematics at this level as a kind of structure.

To succeed in mathematics we need to consider several different notions:

- Technical terms that we understand to be true, but are unable to define exactly without resorting to a circular argument (using the idea in its definition) are called *undefined terms*. Undefined terms may be used as arguments in proofs, but there is the risk that such ambigious terms will lead to unclear proofs.
- A statement that is either true of false is called a *proposition*. Propositions that contain only one part is called an *atomic proposition*. Propositions containing several parts are called *compound propositions*. Usually we are trying to prove a proposition.
- Propositions that we assume to be true based on experience are called *axioms* or *postulates*. Axioms and postulates may be used as arguments in proofs.
- Propositions that we believe to be true, but have not been proved are called *conjectures*. Conjectures may be used as arguments in proofs, but the proof will be undone should a conjecture be disproved.
- A conjecture that has been proved is called a *theorem*. A theorem that is proven as part of a larger proof (as an intermediate step) is called a *lemma*. A theorem that is a minor extension of another theorem is called a *corollary*. Theorems, lemmas, and corollaries may be used as arguments in a proof.
- Technical terms that are built out of precise statements are called *formal definitions*, or just *definitions*. Definitions may be used as arguments in a proof.

Propositional Logic

In the table below you will find definitions and examples of the operations of the logic of propositions. It will be understood that a proposition will be symbolized as $p, q, r, s, \ldots$. All of these symbols may be used in proofs.

Propositional Operation	Symbol	Meaning	Eaxample
Negation	$\neg$	Not	$\neg p$
Conjunction	$\bigwedge$	This And That	$p \wedge q$
Disjunction	$\bigvee$	This Or That	$p \vee q$

Exclusive Disjunction	$\veebar$	This Or That But Not Both	$p \veebar q$
Conditional	$\Rightarrow$	If p, Then q	$p \Longrightarrow q$
Converse	$\Rightarrow$	The Converse of If p, Then q is If q, Then p.	$q \Longrightarrow p$
Contrapositive	$\Rightarrow$	The Contrapositive of If p, Then q is If Not $-\ q$, Then Not $-\ p$.	$\neg q \Longrightarrow \neg p$
Bicondictional	$\Leftrightarrow$	p If and Only If q. If and only if, is sometimes written iff.	$p \Leftrightarrow q$

From these symbols we can create logical formulas. The simplest formula is just the statement of a proposition, for example p, or if we are making a statement that a proposition p depends on another idea, say x we would write $p(x)$.

Truth Tables

Every proposition, indeed every logical formula, is either true or false. We can create a table of these values using T for true, and F for false. When we make this array using all possible truth values, we call it a *truth table*. For example, we can create the truth table for the negation of a proposition p:

p	$\neg p$
T	F
F	T

Here is the truth table for the conjunction between two propositions p and q, where we list all possible truth values of the propositions and apply the definition of the conjunction to determine the resulting truth value.

p	q	$p \wedge q$
T	T	T
F	T	F
T	F	F
F	F	F

Here is the truth table for a somewhat complicated formula:

p	q	r	$p \wedge q$	$\neg r$	$(p \wedge q) \vee \neg r$
T	T	T	T	F	T
F	T	T	F	F	F
T	T	F	T	T	T
F	T	F	F	T	T
T	F	T	F	F	F
F	F	T	F	F	F
T	F	F	F	T	T
F	F	F	F	T	T

If two formulas have the same truth table result, then they are said to be *logically equivalent*. We write $p \sim q$ if p and q are logically equivalent. If a formula is always true, then it is called a *tautology*. If a formula is always false, then it is called a *contradiction*.

Basic Set Theory

The language of modern mathematics is a combination of logic and set theory. We understand a set to be a collection of objects of some kind. Here is a table of basic ideas from set theory.

Idea	Symbol	Meaning
Element of a Set	$x \in X$	x is an element of the set X.
Subset of a Set	$X \subseteq Y$	The set X is a subset of the set Y if every element of X is also an element of Y.
Equal Sets	$X = Y$	The set X is equal to the set Y if every element of X is also an element of Y and every element of Y is also an element of X.
Unequal Sets	$X \neq Y$	X and Y are not equal.
Proper Subset	$X \subset Y$	$X \subseteq Y$ and $X \neq Y$.

Predicate Logic

Not all mathematical statements are propositions. Indeed, $\sqrt{\frac{x}{2}} = 0$, is neither true nor false as it is presented. It becomes a proposition only if we define x in some way. We need to develop a couple of additional ideas.

- A symbol that represents an unspecified object that can be chosen from some set of objects is called a *variable*.
- A statement containing one or more variables that becomes a proposition when the variables are chosen is called a *predicate*.
- The statement, "For every ...," is symbolized by ∀, and is called the *universal quantifier*. For example we can say that for all real numbers, symbolized by ℝ, $x^2 \geq 0$. We could also write $(\forall\, x)\ (x \in \mathbb{R})\ x^2 \geq 0$.
- The statement, "There exists...," is symbolized by ∃, and is called the *existential quantifier*. For example, we can say that there exists some real number such that $x^2 \geq 0$. We could also write $(\exists\, x)\ (x \in \mathbb{R})\ x^2 \geq 0$.

Proof Methods

In what follows, we will identify the starting proposition as the hypothesis and symbolize it by p. The conjecture to be proved, the conclusion, will be symbolized by q.

Proof by Truth Table

This is the most rudimentary style of proof. The primary limitation is the amount of work it requires, and the ever-expanding size of the resulting truth table. You begin by producing the truth table for the hypothesis, and then the conclusion; if they are the same, then they are logically equivalent, thus the hypothesis iff the conclusion.

Direct proof

This is at once the most effective proof and the most difficult. Here are the steps:

1. State the hypothesis.
2. Make your first argument in a sequence that will bring you to the conclusion.
3. ⋮ (this symbol indicates a variable number of steps).
4. Make you final argument.
5. State your conclusion.

Often this process is ended by writing Q.E.D. standing for *qoud erat demonstratum*, meaning roughly, "Which was to be demonstrated."

Proof by contrapositive

The contrapositive and the conditional are logically equivalent, thus if we can prove the contrapositive, we have proven the conditional. We begin this method of proof by stating the conclusion.

1. State the conclusion.
2. Write the negation of the conclusion.
3. Make your first argument in a sequence that will bring you to the hypothesis.
4. ⋮.
5. Make your final argument.
6. State the negation of the hypothesis.
7. Make the argument that by the contrapositive the conditional must be true. Q.E.D.

Reductio ad absurdum (RAA)

Here we assume the negation of the conclusion and show that this leads to a contradiction, thus the negation cannot be true; thus the conclusion must be true:

1. State the hypothesis.
2. Assume that the hypothesis implies the negation of the conclusion
3. Make your first argument in a sequence that will show a contradiction.
4. ⋮.
5. Make you final argument.
6. Show that this implies that the negation of the conclusion is both true and false, such a situation is always false.
7. Since this a contradiction, the negation of the conclusion cannot be true.
8. The conclusion must then be true. Q.E.D.

Mathematical induction

This requires knowing that the natural numbers are 1, 2, 3, and so on.

1. State the hypothesis.
2. Show that the conclusion is true for the case of a variable equal to one. This is called the *basis step*.
3. Write your conclusion for the variable having an arbitrary value for some unspecified natural number n.
4. Show that if the conclusion is true for n that the conclusion is also true for $n + 1$. This is called the *inductive step*. It is possible to reverse 3 and 4, to assume the conclusion true for $n + 1$ and then show that it is true for n.
5. By the Principle of Mathematical Induction the conclusion must be true for all natural numbers (or for all cases that can be listed by the natural numbers). Q.E.D.

Proof by cases - divide and conquer

The final style of proof is given in the next two sections:

1. State the hypothesis.
2. Show that the conclusion requires a finite number of cases.
3. Prove each case independently.
4. Thus the conclusion is true for each possible case. Q.E.D.

Proof by cases - Bootstrap

We continue with the second method for case analysis:

1. State the hypothesis.
2. Show that the conclusion requires a finite number of cases.
3. Prove the first case.
4. Prove each case based on the proof of the previous case.
5. Thus the conclusion is true for each case. Q.E.D.

Counterexamples

Up to now we have considered how to construct a mathematical proof. We can also disprove a conjecture by showing a single case where the conclusion is not true. Such an instance is called a *counterexample* of the conjecture.

Arguments by logic

The following are arguments of logic. It is a useful exercise to prove each of these, either by writing their truth tables, or by other methods.

Argument	Name	Formula	Explanation
1	Contradiction	$(p \wedge \neg p) \Longrightarrow \mathrm{F}$	A proposition and its negation cannot both be true.
2	Double Negative	$\neg(\neg p) \Longrightarrow p$	The negation of a negation of a proposition is the proposition.

3	Law of the Excluded Middle	$(p \vee \neg p)$	Either something is true or it is not. This is similar to argument 1.
4	Commutation	$(p * q) \Longleftrightarrow (q * p)$	This is true when you replace $*$ with either $\wedge$ or $\vee$.
5	Associativity	$(p*q)*r \Longleftrightarrow p*(q*r)$	This is true when you replace $*$ with either $\wedge$ or $\vee$.
6	Law of the Contrapositive	$(p \Longrightarrow q) \Longleftrightarrow (\neg q \Longrightarrow \neg p)$	This is the basis for proof by contrapositive.
7	DeMorgan's Laws	$\neg (p * q) \Longleftrightarrow (\neg p \circ \neg q)$	This is true when you replace $*$ with either $\wedge$ or $\vee$ and $\circ$ with either $\vee$ or $\wedge$, respectively.
8	Distribution	$p*(q \circ r) \Longleftrightarrow (p*q) \circ (p*r)$	This is true when you replace $*$ with either $\wedge$ or $\vee$ and $\circ$ with either $\vee$ or $\wedge$, respectively.

Proof: Here is an example of a proof by truth table, we will prove Argument 1, Contradiction.

p	$\neg p$	$(p \wedge \neg p)$	Contradiction
T	F	F	F
F	T	F	F

thus $(p \wedge \neg p) \sim$ Contradiction, which proves argument 1. QED.

Here is an example of how to discover a proof. We will prove Argument 2, Double Negative. We need to show that the double negative is equivalent to the initial proposition.

1. We start by stating that the negation of a proposition always has the opposite truth value of a proposition, thus we can write

$q = \neg p.$

2. The negation of q will then have the opposite truth value from q, we can write,

$r = \neg q.$

3. Since a proposition is either true or false, when a negation is false the starting proposition is true.

4. When r is false, then q must be true, this also means that p is false.

5. Similarly when r is true q is false, and thus p is true.

6. Therefore we see that r and p are the same.

7. Since r is the double negative of p, then we can say that the double negative of any proposition is the same as the proposition. This has been a proof by cases. QED.

Conclusions

I have presented a fairly good reference for beginning to explore the mathematics used in physics. This is a good beginning.

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