Tensors in *Mathematica* 9: Built-In Capabilities

George E. Hrabovsky
MAST

This Talk

I intend to cover four main topics:

- **How to make tensors in the newest version of *Mathematica*.**
- **The metric tensor and how to transform vectors into covectors.**
- **Cartesian tensor operations.**
- **GR Operations**

How to Build a Tensor in *Mathematica* 9

Rank One

For rank one tensors, we can write them as tangent vectors,

```
Table[{x^i}, {i, {"1", "2", "3"}}] // MatrixForm
```

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

or as covectors

```
Table[x_i, {i, {"1", "2", "3"}}]
```

$\{x_1, x_2, x_3\}$

Rank Two and Higher

We can use similar methods to develop rank two tensors, though *Mathematica* is not able to cope with abstract indices without help from third-party software—I like xAct.

```
Table[σ^(i j), {i, {x, y, z}}, {j, {x, y, z}}] // MatrixForm
```

$$\begin{pmatrix} \sigma^{x^2} & \sigma^{x\,y} & \sigma^{x\,z} \\ \sigma^{x\,y} & \sigma^{y^2} & \sigma^{y\,z} \\ \sigma^{x\,z} & \sigma^{y\,z} & \sigma^{z^2} \end{pmatrix}$$

```
Table[σ^(Grid[{{i,j}},Spacings→.1]), {i, {x, y, z}}, {j, {x, y, z}}] // MatrixForm
```

$$\begin{pmatrix} \sigma^{x\,x} & \sigma^{x\,y} & \sigma^{x\,z} \\ \sigma^{y\,x} & \sigma^{y\,y} & \sigma^{y\,z} \\ \sigma^{z\,x} & \sigma^{z\,y} & \sigma^{z\,z} \end{pmatrix}$$

You can produce the individual tensor components,

```
σ[i_, j_, n_] :=
 Table[(η (∂_(x_j) v_i[x_j] + ∂_(x_i) v_j[x_i] - If[i == j, 2/3 ∂_(x_k) v_k[x_k], 0]) + If[i == j, ξ ∂_(x_k) v_k[x_k], 0]),
   {k, 1, n, 1}] // MatrixForm
```

```
{σ[1, 1, 3], σ[1, 2, 3], σ[1, 3, 3]} // TraditionalForm
```

$$\left\{\begin{pmatrix} \frac{4}{3}\eta\, v_1'(x_1) + \xi\, v_1'(x_1) \\ \eta\left(2\, v_1'(x_1) - \frac{2}{3}\, v_2'(x_2)\right) + \xi\, v_2'(x_2) \\ \eta\left(2\, v_1'(x_1) - \frac{2}{3}\, v_3'(x_3)\right) + \xi\, v_3'(x_3) \end{pmatrix}, \begin{pmatrix} \eta\,(v_1'(x_2) + v_2'(x_1)) \\ \eta\,(v_1'(x_2) + v_2'(x_1)) \\ \eta\,(v_1'(x_2) + v_2'(x_1)) \end{pmatrix}, \begin{pmatrix} \eta\,(v_1'(x_3) + v_3'(x_1)) \\ \eta\,(v_1'(x_3) + v_3'(x_1)) \\ \eta\,(v_1'(x_3) + v_3'(x_1)) \end{pmatrix}\right\}$$

You can also write a table to produce the entire tensor

```
st[n_] := Table[σ[i, j, n], {i, 1, n, 1}, {j, 1, n, 1}] // MatrixForm
st[3] // TraditionalForm
```

$$\begin{pmatrix} \begin{pmatrix} \frac{4}{3}\eta\, v_1'(x_1) + \xi\, v_1'(x_1) \\ \eta\left(2\, v_1'(x_1) - \frac{2}{3}\, v_2'(x_2)\right) + \xi\, v_2'(x_2) \\ \eta\left(2\, v_1'(x_1) - \frac{2}{3}\, v_3'(x_3)\right) + \xi\, v_3'(x_3) \end{pmatrix} & \begin{pmatrix} \eta\,(v_1'(x_2) + v_2'(x_1)) \\ \eta\,(v_1'(x_2) + v_2'(x_1)) \\ \eta\,(v_1'(x_2) + v_2'(x_1)) \end{pmatrix} & \begin{pmatrix} \eta\,(v_1'(x_3) + v_3'(x_1)) \\ \eta\,(v_1'(x_3) + v_3'(x_1)) \\ \eta\,(v_1'(x_3) + v_3'(x_1)) \end{pmatrix} \\ \begin{pmatrix} \eta\,(v_1'(x_2) + v_2'(x_1)) \\ \eta\,(v_1'(x_2) + v_2'(x_1)) \\ \eta\,(v_1'(x_2) + v_2'(x_1)) \end{pmatrix} & \begin{pmatrix} \eta\left(2\, v_2'(x_2) - \frac{2}{3}\, v_1'(x_1)\right) + \xi\, v_1'(x_1) \\ \frac{4}{3}\eta\, v_2'(x_2) + \xi\, v_2'(x_2) \\ \eta\left(2\, v_2'(x_2) - \frac{2}{3}\, v_3'(x_3)\right) + \xi\, v_3'(x_3) \end{pmatrix} & \begin{pmatrix} \eta\,(v_2'(x_3) + v_3'(x_2)) \\ \eta\,(v_2'(x_3) + v_3'(x_2)) \\ \eta\,(v_2'(x_3) + v_3'(x_2)) \end{pmatrix} \\ \begin{pmatrix} \eta\,(v_1'(x_3) + v_3'(x_1)) \\ \eta\,(v_1'(x_3) + v_3'(x_1)) \\ \eta\,(v_1'(x_3) + v_3'(x_1)) \end{pmatrix} & \begin{pmatrix} \eta\,(v_2'(x_3) + v_3'(x_2)) \\ \eta\,(v_2'(x_3) + v_3'(x_2)) \\ \eta\,(v_2'(x_3) + v_3'(x_2)) \end{pmatrix} & \begin{pmatrix} \eta\left(2\, v_3'(x_3) - \frac{2}{3}\, v_1'(x_1)\right) + \xi\, v_1'(x_1) \\ \eta\left(2\, v_3'(x_3) - \frac{2}{3}\, v_2'(x_2)\right) + \xi\, v_2'(x_2) \\ \frac{4}{3}\eta\, v_3'(x_3) + \xi\, v_3'(x_3) \end{pmatrix} \end{pmatrix}$$

The Metric Tensor

A specific example of a calculation is one where we transform from a tangent vector to a covector using the metric tensor,

$$v_i = g_{ij}\, v^j \tag{1}$$

So, given the tangent vector $v^j = (2, r\cos\theta, -r\,\phi)$, and assuming we are in spherical coordinates, we can find the metric.

```
tv = {2, r Cos[θ], -r ϕ};
met =
 CoordinateChartData["Spherical", "Metric", {r, θ, ϕ}] // TraditionalForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2(\theta) \end{pmatrix}$$

We can even find the inverse metric,

```
im = CoordinateChartData["Spherical",
   "InverseMetric", {r, θ, ϕ}] // TraditionalForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{\csc^2(\theta)}{r^2} \end{pmatrix}$$

We take the product of the inverse metric with the tangent vector,

```
tv . CoordinateChartData["Spherical", "InverseMetric", {r, θ, ϕ}] //
 TraditionalForm
```

giving us the covector

$$\left\{2, \frac{\cos(\theta)}{r}, -\frac{\phi \csc^2(\theta)}{r}\right\}$$

The Cartesian Tensor

We create a new tensor,

```
newtensor = Table[x^Grid[{{i}}] y^Grid[{{j}}] z^Grid[{{k}}], {i, 1, 3}, {j, 1, 3}, {k, 1, 3}];
newtensor // MatrixForm // TraditionalForm
```

$$\begin{pmatrix} \begin{pmatrix} x^1 y^1 z^1 \\ x^1 y^1 z^2 \\ x^1 y^1 z^3 \end{pmatrix} & \begin{pmatrix} x^1 y^2 z^1 \\ x^1 y^2 z^2 \\ x^1 y^2 z^3 \end{pmatrix} & \begin{pmatrix} x^1 y^3 z^1 \\ x^1 y^3 z^2 \\ x^1 y^3 z^3 \end{pmatrix} \\ \begin{pmatrix} x^2 y^1 z^1 \\ x^2 y^1 z^2 \\ x^2 y^1 z^3 \end{pmatrix} & \begin{pmatrix} x^2 y^2 z^1 \\ x^2 y^2 z^2 \\ x^2 y^2 z^3 \end{pmatrix} & \begin{pmatrix} x^2 y^3 z^1 \\ x^2 y^3 z^2 \\ x^2 y^3 z^3 \end{pmatrix} \\ \begin{pmatrix} x^3 y^1 z^1 \\ x^3 y^1 z^2 \\ x^3 y^1 z^3 \end{pmatrix} & \begin{pmatrix} x^3 y^2 z^1 \\ x^3 y^2 z^2 \\ x^3 y^2 z^3 \end{pmatrix} & \begin{pmatrix} x^3 y^3 z^1 \\ x^3 y^3 z^2 \\ x^3 y^3 z^3 \end{pmatrix} \end{pmatrix}$$

We can see if its a tensor,

```
ArrayQ[newtensor]
```

```
True
```

We can find its rank

```
TensorRank[newtensor]
```

```
3
```

We can perform a contraction

```
contens = Table[∑_{a=1}^{3} newtensor[[i, a, a]], {i, 1, 3}] // MatrixForm // TraditionalForm
```

$$\begin{pmatrix} x^1 y^1 z^1 + x^1 y^2 z^2 + x^1 y^3 z^3 \\ x^2 y^1 z^1 + x^2 y^2 z^2 + x^2 y^3 z^3 \\ x^3 y^1 z^1 + x^3 y^2 z^2 + x^3 y^3 z^3 \end{pmatrix}$$

We have the inner product

```
in1 = Inner[Times, newtensor, newtensor, Plus] // FullSimplify // MatrixForm
```

$$\begin{pmatrix} \begin{pmatrix} x^1 y^{2\,1} z^1 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^1 y^{2\,1} z^2 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^1 y^{2\,1} z^3 (x^1 z^1 + x^2 z^2 + x^3 z^3) \\ x^1 y^{1+2} z^1 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^1 y^{1+2} z^2 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^1 y^{1+2} z^3 (x^1 z^1 + x^2 z^2 + x^3 z^3) \\ x^1 y^{1+3} z^1 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^1 y^{1+3} z^2 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^1 y^{1+3} z^3 (x^1 z^1 + x^2 z^2 + x^3 z^3) \end{pmatrix} & \begin{pmatrix} x^1 \\ x^1 \\ x^1 \end{pmatrix} \\ \begin{pmatrix} x^2 y^{2\,1} z^1 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^2 y^{2\,1} z^2 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^2 y^{2\,1} z^3 (x^1 z^1 + x^2 z^2 + x^3 z^3) \\ x^2 y^{1+2} z^1 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^2 y^{1+2} z^2 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^2 y^{1+2} z^3 (x^1 z^1 + x^2 z^2 + x^3 z^3) \\ x^2 y^{1+3} z^1 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^2 y^{1+3} z^2 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^2 y^{1+3} z^3 (x^1 z^1 + x^2 z^2 + x^3 z^3) \end{pmatrix} & \begin{pmatrix} x^2 \\ x^2 \\ x^2 \end{pmatrix} \\ \begin{pmatrix} x^3 y^{2\,1} z^1 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^3 y^{2\,1} z^2 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^3 y^{2\,1} z^3 (x^1 z^1 + x^2 z^2 + x^3 z^3) \\ x^3 y^{1+2} z^1 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^3 y^{1+2} z^2 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^3 y^{1+2} z^3 (x^1 z^1 + x^2 z^2 + x^3 z^3) \\ x^3 y^{1+3} z^1 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^3 y^{1+3} z^2 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^3 y^{1+3} z^3 (x^1 z^1 + x^2 z^2 + x^3 z^3) \end{pmatrix} & \begin{pmatrix} x^3 \\ x^3 \\ x^3 \end{pmatrix} \end{pmatrix}$$

```
newtensor . newtensor // FullSimplify // MatrixForm
```

$$\begin{pmatrix} \begin{pmatrix} x^1 y^{2\,1} z^1 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^1 y^{2\,1} z^2 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^1 y^{2\,1} z^3 (x^1 z^1 + x^2 z^2 + x^3 z^3) \\ x^1 y^{1+2} z^1 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^1 y^{1+2} z^2 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^1 y^{1+2} z^3 (x^1 z^1 + x^2 z^2 + x^3 z^3) \\ x^1 y^{1+3} z^1 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^1 y^{1+3} z^2 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^1 y^{1+3} z^3 (x^1 z^1 + x^2 z^2 + x^3 z^3) \end{pmatrix} & \begin{pmatrix} x^1 \\ x^1 \\ x^1 \end{pmatrix} \\ \begin{pmatrix} x^2 y^{2\,1} z^1 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^2 y^{2\,1} z^2 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^2 y^{2\,1} z^3 (x^1 z^1 + x^2 z^2 + x^3 z^3) \\ x^2 y^{1+2} z^1 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^2 y^{1+2} z^2 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^2 y^{1+2} z^3 (x^1 z^1 + x^2 z^2 + x^3 z^3) \\ x^2 y^{1+3} z^1 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^2 y^{1+3} z^2 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^2 y^{1+3} z^3 (x^1 z^1 + x^2 z^2 + x^3 z^3) \end{pmatrix} & \begin{pmatrix} x^2 \\ x^2 \\ x^2 \end{pmatrix} \\ \begin{pmatrix} x^3 y^{2\,1} z^1 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^3 y^{2\,1} z^2 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^3 y^{2\,1} z^3 (x^1 z^1 + x^2 z^2 + x^3 z^3) \\ x^3 y^{1+2} z^1 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^3 y^{1+2} z^2 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^3 y^{1+2} z^3 (x^1 z^1 + x^2 z^2 + x^3 z^3) \\ x^3 y^{1+3} z^1 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^3 y^{1+3} z^2 (x^1 z^1 + x^2 z^2 + x^3 z^3) & x^3 y^{1+3} z^3 (x^1 z^1 + x^2 z^2 + x^3 z^3) \end{pmatrix} & \begin{pmatrix} x^3 \\ x^3 \\ x^3 \end{pmatrix} \end{pmatrix}$$

We can take the direct product,

```
in1 = Outer[Times, newtensor, newtensor] // FullSimplify // MatrixForm
```

A very large output was generated. Here is a sample of it:

$$\begin{pmatrix} \begin{pmatrix} \begin{pmatrix} x^{2\,1} y^{2\,1} z^{2\,1} & x^{2\,1} y^{2\,1} z^{1+2} & x^{2\,1} y^{2\,1} z^{1+3} \\ x^{2\,1} y^{1+2} z^{2\,1} & x^{2\,1} y^{1+2} z^{1+2} & x^{2\,1} y^{\ll 1 \gg} z^{\ll 1 \gg} \\ \ll 1 \gg & \ll 1 \gg & \ll 1 \gg \end{pmatrix} & (\ll 1 \gg) & (\ll 1 \gg) \\ (\ll 1 \gg) & (\ll 1 \gg) & (\ll 1 \gg) \\ (\ll 1 \gg) & (\ll 1 \gg) & (\ll 1 \gg) \end{pmatrix} & \begin{pmatrix} (\ll 1 \gg) & (\ll 1 \gg) & (\ll 1 \gg) \\ (\ll 1 \gg) & (\ll 1 \gg) & (\ll 1 \gg) \\ (\ll 1 \gg) & (\ll 1 \gg) & (\ll 1 \gg) \end{pmatrix} \\ \begin{pmatrix} (\ll 1 \gg) & (\ll 1 \gg) & (\ll 1 \gg) \\ (\ll 1 \gg) & (\ll 1 \gg) & (\ll 1 \gg) \\ (\ll 1 \gg) & (\ll 1 \gg) & (\ll 1 \gg \; \ll 1 \gg) \end{pmatrix} & (\ll 1 \gg) \\ (\ll 1 \gg) & (\ll 1 \gg) \end{pmatrix}$$

Show Less | Show More | Show Full Output | Set Size Limit...

We can take the trace of the tensor,

```
Tr[newtensor]
```

$x^1 y^1 z^1 + x^2 y^2 z^2 + x^3 y^3 z^3$

We can transpose,

```
Transpose[newtensor] // MatrixForm // TraditionalForm
```

$$\begin{pmatrix} \begin{pmatrix} x^1 y^1 z^1 \\ x^1 y^1 z^2 \\ x^1 y^1 z^3 \end{pmatrix} & \begin{pmatrix} x^2 y^1 z^1 \\ x^2 y^1 z^2 \\ x^2 y^1 z^3 \end{pmatrix} & \begin{pmatrix} x^3 y^1 z^1 \\ x^3 y^1 z^2 \\ x^3 y^1 z^3 \end{pmatrix} \\ \begin{pmatrix} x^1 y^2 z^1 \\ x^1 y^2 z^2 \\ x^1 y^2 z^3 \end{pmatrix} & \begin{pmatrix} x^2 y^2 z^1 \\ x^2 y^2 z^2 \\ x^2 y^2 z^3 \end{pmatrix} & \begin{pmatrix} x^3 y^2 z^1 \\ x^3 y^2 z^2 \\ x^3 y^2 z^3 \end{pmatrix} \\ \begin{pmatrix} x^1 y^3 z^1 \\ x^1 y^3 z^2 \\ x^1 y^3 z^3 \end{pmatrix} & \begin{pmatrix} x^2 y^3 z^1 \\ x^2 y^3 z^2 \\ x^2 y^3 z^3 \end{pmatrix} & \begin{pmatrix} x^3 y^3 z^1 \\ x^3 y^3 z^2 \\ x^3 y^3 z^3 \end{pmatrix} \end{pmatrix}$$

We can take a partial derivative

```
∂x newtensor // MatrixForm // TraditionalForm
```

$$\begin{pmatrix} \begin{pmatrix} 1\ x^{1-1} y^1 z^1 \\ 1\ x^{1-1} y^1 z^2 \\ 1\ x^{1-1} y^1 z^3 \end{pmatrix} & \begin{pmatrix} 1\ x^{1-1} y^2 z^1 \\ 1\ x^{1-1} y^2 z^2 \\ 1\ x^{1-1} y^2 z^3 \end{pmatrix} & \begin{pmatrix} 1\ x^{1-1} y^3 z^1 \\ 1\ x^{1-1} y^3 z^2 \\ 1\ x^{1-1} y^3 z^3 \end{pmatrix} \\ \begin{pmatrix} 2\ x^{2-1} y^1 z^1 \\ 2\ x^{2-1} y^1 z^2 \\ 2\ x^{2-1} y^1 z^3 \end{pmatrix} & \begin{pmatrix} 2\ x^{2-1} y^2 z^1 \\ 2\ x^{2-1} y^2 z^2 \\ 2\ x^{2-1} y^2 z^3 \end{pmatrix} & \begin{pmatrix} 2\ x^{2-1} y^3 z^1 \\ 2\ x^{2-1} y^3 z^2 \\ 2\ x^{2-1} y^3 z^3 \end{pmatrix} \\ \begin{pmatrix} 3\ x^{3-1} y^1 z^1 \\ 3\ x^{3-1} y^1 z^2 \\ 3\ x^{3-1} y^1 z^3 \end{pmatrix} & \begin{pmatrix} 3\ x^{3-1} y^2 z^1 \\ 3\ x^{3-1} y^2 z^2 \\ 3\ x^{3-1} y^2 z^3 \end{pmatrix} & \begin{pmatrix} 3\ x^{3-1} y^3 z^1 \\ 3\ x^{3-1} y^3 z^2 \\ 3\ x^{3-1} y^3 z^3 \end{pmatrix} \end{pmatrix}$$

we can determine the differentials

```
Dt[newtensor] // MatrixForm // TraditionalForm
```

$$\left(\begin{array}{ccc}
\begin{pmatrix} dx\ 1\ x^{1-1} y^1 z^1 + dy\ 1\ x^1 y^{1-1} z^1 + dz\ 1\ x^1 y^1 z^{1-1} \\ dx\ 1\ x^{1-1} y^1 z^2 + dy\ 1\ x^1 y^{1-1} z^2 + dz\ 2\ x^1 y^1 z^{2-1} \\ dx\ 1\ x^{1-1} y^1 z^3 + dy\ 1\ x^1 y^{1-1} z^3 + dz\ 3\ x^1 y^1 z^{3-1} \end{pmatrix} & \begin{pmatrix} dx\ 1\ x^{1-1} y^2 z^1 + dy\ 2\ x^1 y^{2-1} z^1 + dz\ 1\ x^1 y^2 z^{1-1} \\ dx\ 1\ x^{1-1} y^2 z^2 + dy\ 2\ x^1 y^{2-1} z^2 + dz\ 2\ x^1 y^2 z^{2-1} \\ dx\ 1\ x^{1-1} y^2 z^3 + dy\ 2\ x^1 y^{2-1} z^3 + dz\ 3\ x^1 y^2 z^{3-1} \end{pmatrix} & \begin{pmatrix} dx\ 1\ x^{1-} \\ dx\ 1\ x^{1-} \\ dx\ 1\ x^{1-} \end{pmatrix} \\
\begin{pmatrix} dx\ 2\ x^{2-1} y^1 z^1 + dy\ 1\ x^2 y^{1-1} z^1 + dz\ 1\ x^2 y^1 z^{1-1} \\ dx\ 2\ x^{2-1} y^1 z^2 + dy\ 1\ x^2 y^{1-1} z^2 + dz\ 2\ x^2 y^1 z^{2-1} \\ dx\ 2\ x^{2-1} y^1 z^3 + dy\ 1\ x^2 y^{1-1} z^3 + dz\ 3\ x^2 y^1 z^{3-1} \end{pmatrix} & \begin{pmatrix} dx\ 2\ x^{2-1} y^2 z^1 + dy\ 2\ x^2 y^{2-1} z^1 + dz\ 1\ x^2 y^2 z^{1-1} \\ dx\ 2\ x^{2-1} y^2 z^2 + dy\ 2\ x^2 y^{2-1} z^2 + dz\ 2\ x^2 y^2 z^{2-1} \\ dx\ 2\ x^{2-1} y^2 z^3 + dy\ 2\ x^2 y^{2-1} z^3 + dz\ 3\ x^2 y^2 z^{3-1} \end{pmatrix} & \begin{pmatrix} dx\ 2\ x^{2-} \\ dx\ 2\ x^{2-} \\ dx\ 2\ x^{2-} \end{pmatrix} \\
\begin{pmatrix} dx\ 3\ x^{3-1} y^1 z^1 + dy\ 1\ x^3 y^{1-1} z^1 + dz\ 1\ x^3 y^1 z^{1-1} \\ dx\ 3\ x^{3-1} y^1 z^2 + dy\ 1\ x^3 y^{1-1} z^2 + dz\ 2\ x^3 y^1 z^{2-1} \\ dx\ 3\ x^{3-1} y^1 z^3 + dy\ 1\ x^3 y^{1-1} z^3 + dz\ 3\ x^3 y^1 z^{3-1} \end{pmatrix} & \begin{pmatrix} dx\ 3\ x^{3-1} y^2 z^1 + dy\ 2\ x^3 y^{2-1} z^1 + dz\ 1\ x^3 y^2 z^{1-1} \\ dx\ 3\ x^{3-1} y^2 z^2 + dy\ 2\ x^3 y^{2-1} z^2 + dz\ 2\ x^3 y^2 z^{2-1} \\ dx\ 3\ x^{3-1} y^2 z^3 + dy\ 2\ x^3 y^{2-1} z^3 + dz\ 3\ x^3 y^2 z^{3-1} \end{pmatrix} & \begin{pmatrix} dx\ 3\ x^{3-} \\ dx\ 3\ x^{3-} \\ dx\ 3\ x^{3-} \end{pmatrix}
\end{array}\right.$$

We can look for symmetries,

```
TensorSymmetry[newtensor]
```

```
{}
```

```
symten = Symmetrize[newtensor, Antisymmetric[{1, 2}]]
```

```
StructuredArray[SymmetrizedArray, {3, 3, 3}, -Structured Data-]
```

```
Normal[symten] // MatrixForm
```

$$\begin{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \frac{1}{2}\left(-x^2 y^1 z^1 + x^1 y^2 z^1\right) \\ \frac{1}{2}\left(-x^2 y^1 z^2 + x^1 y^2 z^2\right) \\ \frac{1}{2}\left(-x^2 y^1 z^3 + x^1 y^2 z^3\right) \end{pmatrix} & \begin{pmatrix} \frac{1}{2}\left(-x^3 y^1 z^1 + x^1 y^3 z^1\right) \\ \frac{1}{2}\left(-x^3 y^1 z^2 + x^1 y^3 z^2\right) \\ \frac{1}{2}\left(-x^3 y^1 z^3 + x^1 y^3 z^3\right) \end{pmatrix} \\ \begin{pmatrix} \frac{1}{2}\left(x^2 y^1 z^1 - x^1 y^2 z^1\right) \\ \frac{1}{2}\left(x^2 y^1 z^2 - x^1 y^2 z^2\right) \\ \frac{1}{2}\left(x^2 y^1 z^3 - x^1 y^2 z^3\right) \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \frac{1}{2}\left(-x^3 y^2 z^1 + x^2 y^3 z^1\right) \\ \frac{1}{2}\left(-x^3 y^2 z^2 + x^2 y^3 z^2\right) \\ \frac{1}{2}\left(-x^3 y^2 z^3 + x^2 y^3 z^3\right) \end{pmatrix} \\ \begin{pmatrix} \frac{1}{2}\left(x^3 y^1 z^1 - x^1 y^3 z^1\right) \\ \frac{1}{2}\left(x^3 y^1 z^2 - x^1 y^3 z^2\right) \\ \frac{1}{2}\left(x^3 y^1 z^3 - x^1 y^3 z^3\right) \end{pmatrix} & \begin{pmatrix} \frac{1}{2}\left(x^3 y^2 z^1 - x^2 y^3 z^1\right) \\ \frac{1}{2}\left(x^3 y^2 z^2 - x^2 y^3 z^2\right) \\ \frac{1}{2}\left(x^3 y^2 z^3 - x^2 y^3 z^3\right) \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix}$$

```
TensorSymmetry[symten]
```

```
Antisymmetric[{1, 2}]
```

GR Operations

The Metric

Here we input the metric

In[41]:=
```
Metric[gi_] := Module[
  {n = Length[gi], g, ing},
  g = Table[If[μ ≥ ν, gi[[μ, ν]], gi[[ν, μ]]], {μ, n}, {ν, n}];
  ing = Simplify[Inverse[g]];
  {n, g, ing}]
```

we assign labels to the coordinates

In[42]:=
```
Evaluate[Table[x[μ], {μ, 4}]] = {t, r, θ, φ};
```

Here is the Schwarzschild metric:

In[43]:=
```
Metric[{{1 - 1 / r}, {0, -1 / (1 - 1 / r)}, {0, 0, -r^2}, {0, 0, 0, -r^2 2 Sin[θ]^2}}] //
  TraditionalForm
```

Out[43]//TraditionalForm=

$$\left\{4, \begin{pmatrix} 1-\frac{1}{r} & 0 & 0 & 0 \\ 0 & -\frac{1}{1-\frac{1}{r}} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -2\,r^2 \sin^2(\theta) \end{pmatrix}, \begin{pmatrix} \frac{r}{r-1} & 0 & 0 & 0 \\ 0 & \frac{1}{r}-1 & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{\csc^2(\theta)}{2\,r^2} \end{pmatrix}\right\}$$

Here we have the Christoffel Symbols:

In[52]:=
```
Christoffel[{n_, g_, ing_}, OptionsPattern[]] := Module[
  {Γ, inΓ}, Γ = inΓ = Table[0, {λ, n}, {μ, n}, {ν, n}];
  Do[
   Γ[[λ, μ, ν]] = Simplify[(D[g[[λ, ν]], x[μ]] + D[g[[λ, μ]], x[ν]] - D[g[[μ, ν]], x[λ]]) / 2];
   If[μ ≠ ν, Γ[[λ, ν, μ]] = Γ[[λ, μ, ν]]], {λ, n}, {μ, n}, {ν, μ}];
  Do[inΓ[[λ, μ, ν]] = Simplify[Sum[ing[[λ, ρ]] Γ[[ρ, μ, ν]], {ρ, n}]];
   If[μ ≠ ν, inΓ[[λ, ν, μ]] = inΓ[[λ, μ, ν]]], {λ, n}, {μ, n}, {ν, μ}];
  If[OptionValue[PrintNonZero],
   Do[If[Γ[[λ, μ, ν]] =!= 0, Print["Γ"_Grid[{{λ-1,μ-1,ν-1}}], " ", Γ[[λ, μ, ν]]]],
    {λ, n}, {μ, n}, {ν, μ}]; Do[If[inΓ[[λ, μ, ν]] =!= 0,
     Print["Γ"^Grid[{{λ-1}}]_Grid[{{"",μ-1,ν-1}}], " ", inΓ[[λ, μ, ν]]]], {λ, n}, {μ, n}, {ν, μ}
   ]];
  {Γ, inΓ}]
```

In[49]:=
```
Christoffel[{n_, g_, ing_}, OptionsPattern[]] := Module[
  {Γ, inΓ}, Γ = inΓ = Table[0, {λ, n}, {μ, n}, {ν, n}];
  Do[
   Γ[[λ, μ, ν]] = Simplify[(D[g[[λ, ν]], x[μ]] + D[g[[λ, μ]], x[ν]] - D[g[[μ, ν]], x[λ]]) / 2];
   If[μ ≠ ν, Γ[[λ, ν, μ]] = Γ[[λ, μ, ν]]], {λ, n}, {μ, n}, {ν, μ}];
  Do[inΓ[[λ, μ, ν]] = Simplify[Sum[ing[[λ, ρ]] * Γ[[ρ, μ, ν]], {ρ, n}]];
   If[μ ≠ ν, inΓ[[λ, ν, μ]] = inΓ[[λ, μ, ν]]], {λ, n}, {μ, n}, {ν, μ}];
  Do[If[Γ[[λ, μ, ν]] =!= 0, Print["Γ", λ - 1, μ - 1, ν - 1, " ", Γ[[λ, μ, ν]]]],
   {λ, n}, {μ, n}, {ν, μ}]; Do[If[inΓ[[λ, μ, ν]] =!= 0,
    Print["Γ", λ - 1, μ - 1, ν - 1, " ", inΓ[[λ, μ, ν]]]], {λ, n}, {μ, n}, {ν, μ}
  ];
  {Γ, inΓ}]
```

Here we calculate the Christoffel symbols for the Schwarzschild Metric

In[53]:=
```
Christoffel[
  Metric[{{1 - 1 / r}, {0, -1 / (1 - 1 / r)}, {0, 0, -r^2}, {0, 0, 0, -r^2 Sin[θ]^2}}]];
```

$\Gamma_{0\,1\,0}$ $\frac{1}{2\,r^2}$

$\Gamma_{1\,0\,0}$ $-\frac{1}{2\,r^2}$

$\Gamma_{1\,1\,1}$ $\frac{1}{2\,(-1+r)^2}$

$\Gamma_{1\,2\,2}$ r

$\Gamma_{1\,3\,3}$ $r\,\mathrm{Sin}[\theta]^2$

$\Gamma_{2\,2\,1}$ $-r$

$\Gamma_{2\,3\,3}$ $r^2\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]$

$\Gamma_{3\,3\,1}$ $-r\,\mathrm{Sin}[\theta]^2$

$\Gamma_{3\,3\,2}$ $-r^2\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]$

$\Gamma^{0}{}_{1\,0}$ $\frac{1}{2\,(-1+r)\,r}$

$\Gamma^{1}{}_{0\,0}$ $\frac{-1+r}{2\,r^3}$

$\Gamma^{1}{}_{1\,1}$ $\frac{1}{2\,r-2\,r^2}$

$\Gamma^{1}{}_{2\,2}$ $1-r$

$\Gamma^{1}{}_{3\,3}$ $-(-1+r)\,\mathrm{Sin}[\theta]^2$

$\Gamma^{2}{}_{2\,1}$ $\frac{1}{r}$

$\Gamma^{2}{}_{3\,3}$ $-\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]$

$\Gamma^{3}{}_{3\,1}$ $\frac{1}{r}$

$\Gamma^{3}{}_{3\,2}$ $\mathrm{Cot}[\theta]$

Here we calculate the components of the Riemann tensor:

```
In[54]:= Riemann[{n_, g_, ing_}, OptionsPattern[]] := Module[
    {Γ, inΓ, R = Table[0, {α, n}, {β, n}, {μ, n}, {ν, n}], R2 = Table[0, {μ, n}, {ν, n}], R0},
    {Γ, inΓ} = Christoffel[{n, g, ing}]; Do[R[[α, β, μ, ν]] = R[[β, α, ν, μ]] =
        Simplify[Sum[g[[α, λ]] (D[inΓ[[λ, β, ν]], x[μ]] - D[inΓ[[λ, β, μ]], x[ν]]) +
            Γ[[α, λ, μ]] inΓ[[λ, β, ν]] - Γ[[α, λ, ν]] inΓ[[λ, β, μ]], {λ, n}]];
      R[[β, α, μ, ν]] = R[[α, β, ν, μ]] = -R[[α, β, μ, ν]];
      If[μ ≠ α, R[[μ, ν, α, β]] = R[[ν, μ, β, α]] = R[[α, β, μ, ν]];
        R[[ν, μ, α, β]] = R[[μ, ν, β, α]] = -R[[α, β, μ, ν]]],
      {α, 2, n}, {β, α - 1}, {μ, 2, α}, {ν, If[μ === α, β, μ - 1]}];
    Do[R2[[μ, ν]] = Simplify[Sum[ing[[α, β]] * R[[α, μ, β, ν]], {α, n}, {β, n}]];
      If[μ ≠ ν, R2[[ν, μ]] = R2[[μ, ν]]], {μ, n}, {ν, μ}];
    R0 = Simplify[Sum[ing[[μ, ν]] R2[[μ, ν]] If[μ ≠ ν, 2, 1], {μ, n}, {ν, μ}]];
    Do[If[R[[α, β, μ, ν]] =!= 0, Print["R"Grid[{{α-1,β-1,μ-1,ν-1}}], " ", R[[α, β, μ, ν]]]],
      {α, 2, n}, {β, α - 1}, {μ, 2, α}, {ν, If[μ === α, β, μ - 1]}];
    Do[If[R2[[μ, ν]] =!= 0, Print["R"Grid[{{μ-1,ν-1}}], " ", R2[[μ, ν]]]], {μ, n}, {ν, μ}];
    If[R0 =!= 0, Print["R ", R0]];
    {R, R2, R0}]
```

Here is the output for the Schwarzschild metric, assuming $r_S = 1$:

```
In[55]:= Riemann[Metric[{{1 - 1 / r}, {0, -1 / (1 - 1 / r)}, {0, 0, -r^2}, {0, 0, 0, -r^2 Sin[θ]^2}}]];
```

$\Gamma_{0\ 1\ 0}$ $\frac{1}{2\ r^2}$

$\Gamma_{1\ 0\ 0}$ $-\frac{1}{2\ r^2}$

$\Gamma_{1\ 1\ 1}$ $\frac{1}{2\ (-1+r)^2}$

$\Gamma_{1\ 2\ 2}$ r

$\Gamma_{1\ 3\ 3}$ $r\ \mathrm{Sin}[\theta]^2$

$\Gamma_{2\ 2\ 1}$ $-r$

$\Gamma_{2\ 3\ 3}$ $r^2\ \mathrm{Cos}[\theta]\ \mathrm{Sin}[\theta]$

$\Gamma_{3\ 3\ 1}$ $-r\ \mathrm{Sin}[\theta]^2$

$\Gamma_{3\ 3\ 2}$ $-r^2\ \mathrm{Cos}[\theta]\ \mathrm{Sin}[\theta]$

$\Gamma^{0}{}_{1\ 0}$ $\frac{1}{2\ (-1+r)\ r}$

$\Gamma^{1}{}_{0\ 0}$ $\frac{-1+r}{2\ r^3}$

$\Gamma^{1}{}_{1\ 1}$ $\frac{1}{2\ r-2\ r^2}$

$\Gamma^{1}{}_{2\ 2}$ $1-r$

$\Gamma^{1}{}_{3\ 3}$ $-(-1+r)\ \mathrm{Sin}[\theta]^2$

$\Gamma^{2}{}_{2\ 1}$ $\frac{1}{r}$

$\Gamma^{2}{}_{3\ 3}$ $-\mathrm{Cos}[\theta]\ \mathrm{Sin}[\theta]$

$\Gamma^{3}{}_{3\ 1}$ $\frac{1}{r}$

$\Gamma^{3}{}_{3\ 2}$ $\mathrm{Cot}[\theta]$

$R_{1\ 0\ 1\ 0}$ $\frac{1}{r^3}$

$R_{2\ 0\ 2\ 0}$ $-\frac{-1+r}{2\ r^2}$

$R_{2\ 1\ 2\ 1}$ $\frac{1}{2\ (-1+r)}$

$R_{3\ 0\ 3\ 0}$ $-\frac{(-1+r)\ \mathrm{Sin}[\theta]^2}{2\ r^2}$

$R_{3\ 1\ 3\ 1}$ $\frac{\mathrm{Sin}[\theta]^2}{-2+2\ r}$

$R_{3\ 2\ 3\ 2}$ $-r\ \mathrm{Sin}[\theta]^2$

Thank You!

George E. Hrabovsky
MAST
www.madscitech.org
george@madscitech.org