Day 2: Logic and Proof

George E. Hrabovsky
MAST

Introduction

This is the second installment of the series. Here I intend to present the ideas and methods of proof.

Logic and proof

To begin with, I will need to present the basic method of mathematics. I will try to make this as simple as possible and still be useful. It is important to realize that in mathematics, until an idea is applied to something concrete, ideas have no meaning. Thus mathematics is the ultimate abstraction from reality; we speak of pure ideas without regard to meaning. It is best to think of mathematics at this level as a kind of structure.

To succeed in mathematics we need to consider several different notions:

- Technical terms that we understand to be true, but are unable to define exactly without resorting to a circular argument (using the idea in its definition) are called *undefined terms*. Undefined terms may be used as arguments in proofs, but there is the risk that such ambigious terms will lead to unclear proofs.
- A statement that is either true of false is called a *proposition*. Propositions that contain only one part is called an *atomic proposition*. Propositions containing several parts are called *compound propositions*.
- Propositions that we assume to be true based on experience are called *axioms* or *postulates*. Axioms and postulates may be used as arguments in proofs.
- Propositions that we believe to be true, but have not been proved are called *conjectures*. Conjectures may be used as arguments in proofs, but the prove will be undone should a conjecture be disproved.
- A conjecture that has been proved is called a *theorem*. A theorem that is proven as part of a larger proof (as an intermediate step) is called a *lemma*. A theorem that is a minor extension of another theorem is called a *corollary*. Theorems, lemmas, and corollaries may be used as arguments in a proof.

Technical terms that are built out of precise statements are called *formal definitions*, or just *definitions*. Definitions may be used as arguments in a proof.

Propositional Logic

In the table below you will find definitions and examples of the operations of the logic of propositions. It will be understood that a proposition will be symbolized as $p,\ q,\ r,\ s,\ \ldots.$ All of these symbols may be used in proofs.

Propositional Operation	**Symbol**	**Meaning**	**Eaxample**
Negation	$\neg$	Not	$\neg\, p$
Conjunction	$\bigwedge$	This And That	$p \wedge q$
Disjunction	$\bigvee$	This Or That	$p \vee q$
Exclusive Disjunction	$\underline{\bigvee}$	This Or That But Not Both	$p \veebar q$
Conditional	$\Rightarrow$	If p, Then q	$p \Longrightarrow q$
Converse	$\Rightarrow$	The Converse of If p, Then q is If q, Then p.	$q \Longrightarrow p$
Contrapositive	$\Rightarrow$	The Contrapositive of If p, Then q is If Not $-\, q$, Then Not $-\, p$.	$\neg\, q \Longrightarrow \neg\, p$
Bicondictional	$\Leftrightarrow$	p If and Only If q. If and only if, is sometimes written iff.	$p \Leftrightarrow q$

From these symbols we can create logical formulas. The simplest formula is just the statement of a proposition, for example p, or if we are making a statement that a proposition p depends on another idea, say x we would write $p(x)$.

Truth Tables

Every proposition, indeed every logical formula, is either true or false. We can create a table of these values using T for true, and F for false. When we make this array using all possible truth values, we call it a *truth table*. For example, we can create the truth table for the negation of a proposition p:

p	$\neg p$
T	F
F	T

Here is the truth table for the conjunction between two propositions p and q, where we list all possible truth values of the propositions and apply the definition of the conjunction to determine the resulting truth value.

p	q	$p \wedge q$
T	T	T
F	T	F
T	F	F
F	F	F

Here is the truth table for a somewhat complicated formula:

p	q	r	$p \wedge q$	$\neg r$	$(p \wedge q) \vee \neg r$
T	T	T	T	F	T
F	T	T	F	F	F
T	T	F	T	T	T
F	T	F	F	T	T
T	F	T	F	F	F
F	F	T	F	F	F
T	F	F	F	T	T
F	F	F	F	T	T

If two formulas have the same truth table result, then they are said to be logically equivalent. We would write $p \sim q$ if p and q are logically equivalent. If a formula is always true, then it is called a *tautology*. If a formula is always false, then it is called a *contradiction*.

Basic Set Theory

The language of modern mathematics is a combination of logic and set theory. We understand a set to be a collection of objects of some kind. Here is a table of basic ideas from set theory.

Idea	**Symbol**	**Meaning**
Element of a Set	$x \in X$	x is an element of the set X.
Subset of a Set	$X \subseteq Y$	The set X is a subset of the set Y if every element of X is also an element of Y.
Equal Sets	$X = Y$	The set X is equal to the set Y if every element of X is also an element of Y and every element of Y is also an element of X.
Unequal Sets	$X \neq Y$	X and Y are not equal.
Proper Subset	$X \subset Y$	$X \subseteq Y$ and $X \neq Y$.

Predicate Logic

Not all mathematical statements are propositions. Indeed, $\sqrt{\frac{x}{2}} = 0$, is neither true nor false as it is presented. It becomes a proposition only if we define x in some way. We need to develop a couple of additional ideas.

- A symbol that represents an unspecified object that can be chosen from some collection of objects is called a *variable*.
- A statement containing one or more variables that becomes a proposition when the variables are chosen is called a *predicate*.
- The statement, "For every ...," is symbolized by $\forall$, and is called the *universal quantifier*. For example we can say that for all real numbers, symbolized by $\mathbb{R}$, $x^2 \geq 0$. We could also write $(\forall\ x)\ (x \in \mathbb{R})\ x^2 \geq 0$.
- The statement, "There exists...," is symbolized by $\exists$, and is called the *existential quantifier*. For example, we can say that there exists some real number such that $x^2 \geq 0$. We could also write $(\exists\ x)\ (x \in \mathbb{R})\ x^2 \geq 0$.

Proof Methods

In what follows, we will identify the starting proposition, the given, as the hypothesis and symbolize it by p. The conjecture to be proved, the conclusion, will be symbolized by q.

Proof by Truth Table

This is the most rudimentary style of proof. The primary limitation is the amount of work it requires, and the ever-expanding size of the resulting truth table. You begin by producing the truth table for the hypothesis, and then the conclusion; if they are the same, then they are logically equivalent, thus the hypothesis iff the conclusion.

Direct proof

This is at once the most effective proof and the most difficult. Here are the steps:

1. State the hypothesis.
2. Make your first argument in a sequence that will bring you to the conclusion.
3. ⋮ (this symbol indicates a variable number of steps).
4. Make you final argument.
5. State your conclusion.

Often this process is ended by writing Q.E.D. standing for *qoud erat demonstratum*, meaning roughly, "Which was to be demonstrated."

Proof by contrapositive

The contrapositive and the conditional are logically equivalent, thus if we can prove the contrapositive, we have proven the conditional. We begin this method of proof by stating the conclusion.

1. State the conclusion.
2. Write the negation of the conclusion.
3. Make your first argument in a sequence that will bring you to the hypothesis.
4. ⋮.
5. Make your final argument.
6. State the negation of the hypothesis.
7. Make the argument that by the contrapositive the conditional must be true. Q.E.D.

Reductio ad absurdum (RAA)

I gave Galileo's example of this type of proof in the Day One Theoretical Physics Article.

1. State the hypothesis.
2. Assume that the hypothesis implies the negation of the conclusion
3. Make your first argument in a sequence that will bring you to the conclusion.
4. ⋮.
5. Make you final argument.
6. Show that this implies that the negation of the conclusion is both true and false, such a situation is always false.
7. Since this a contradiction, the negation of the conclusion cannot be true.
8. The conclusion must then be true. Q.E.D.

Mathematical induction

This requires knowing that the natural numbers are 1, 2, 3, and so on.

1. State the hypothesis.
2. Show that the conclusion is true for the case of a variable equal to one. This is called the *basis step*.
3. Write your conclusion for the variable having an arbitrary value for some unspecified natural number n.
4. Show that if the conclusion is true for n that the conclusion is also true for $n+1$. This is called the *inductive step*. It is possible to reverse 3 and 4, to assume the conclusion true for $n+1$ and then show that it is true for n.
5. By the Principle of Mathematical Induction the conclusion must be true for all natural numbers (or for all cases that can be listed by the natural numbers). Q.E.D.

Proof by cases - divide and conquer

The final style of proof is given in the next two sections:

1. State the hypothesis.
2. Show that the conclusion requires a finite number of cases.
3. Prove each case independently.
4. Thus the conclusion is true for each possible case. Q.E.D.

Proof by cases - Bootstrap

We continue with the second method for case analysis:

1. State the hypothesis.
2. Show that the conclusion requires a finite number of cases.
3. Prove the first case.
4. Prove each case based on the proof of the previous case.
5. Thus the conclusion is true for each case. Q.E.D.

Counterexamples

Up to now we have considered how to construct a mathematical proof. We can also disprove a conjecture by showing a single case where the conclusion is not true. Such an instance is called a *counterexample* of the conjecture.

Logical Operations in *Mathematica*

Using [1] as a basis, I will begin with some logical operations.

Operation	*Mathematica* Command	Explanation
Negation	! expression	Negates the expression.

Conjunction	e1 && e2 && ...	Returns True if e1 and e2 are true, otherwise it returns False.
Disjunction	e1 || e2 || ...	Returns True if e1 or e2 are true, otherwise it returns False.
Exclusive Disjunction	Xor[e1, e2, ...]	Returns True if either e1 or e2 are true, but not both, otherwise it returns false.
Conditional	Implies[p, q]	This represents the conditional $p \Rightarrow q$.
Biconditional	Equivalent[p, q]	This represents the biconditional $p \Leftrightarrow q$.
ForAll	ForAll[x, expr]	This is the universal quantifier.
Exists	Exists[x, expr]	This is the existential quantifier.

We can use *Mathematica* to develop truth tables. We will first use the command BooleanTable[logical expression,{logical variable 1},{logical variable 2}, ...]

```
BooleanTable[p && q, {p}, {q}]
{{True, False}, {False, False}}
```

We can put this into the form of a table by either wrapping the function in TableForm[],

```
TableForm[BooleanTable[p && q, {p}, {q}]]
True  False
False False
```

or by adding //TableForm on the end

```
BooleanTable[p && q, {p}, {q}] // TableForm
True  False
False False
```

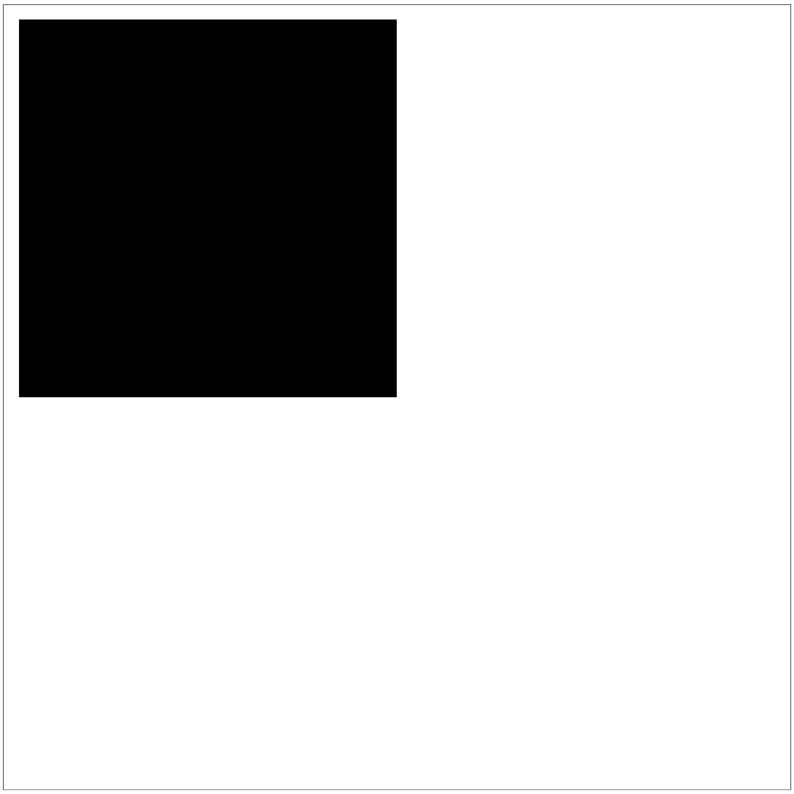
We can make it numerical, where 1 stands for True and 0 for False by Wrapping the command in Boole[ ].

```
Boole[BooleanTable[p && q, {p}, {q}]] // TableForm
```

```
1 0
0 0
```

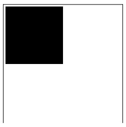
We can even make a pictogram of the truth table by wrapping the command in ArrayPlot. Here the black squares represent the value True and the white the value False.

```
ArrayPlot[Boole[BooleanTable[p && q, {p}, {q}]]]
```



We can make the image smaller by specifying an ImageSize->45

```
ArrayPlot[Boole[BooleanTable[p && q, {p}, {q}]],
 ImageSize → 45]
```



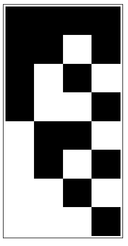
Here is the picture of a truth table for a more complicated formula,

```
BooleanTable[{p, q, r, (p && q) || (! r)}] //
 TableForm
```

```
True  True  True  True
True  True  False True
True  False True  False
True  False False True
False True  True  False
False True  False True
False False True  False
False False False True
```

Or graphically,

```
ArrayPlot[
 Boole[BooleanTable[{p, q, r, (p && q) || (! r)}]],
 ImageSize → 45]
```



In this way it is easy to see logical equivalence and to use truth tables to prove logical statements.

Arguments by logic

The following are arguments of logic. It is a useful exercise to prove each of these, either by writing their truth tables, or by other methods.

Argument	Name	Formula	Explanation
1	Definition of a Contradiction	$(p \wedge \neg\, p) \Longrightarrow F$	A proposition and its negation cannot both be true.
2	Definition of a Double Negative	$\neg\,(\neg\, p) \Longrightarrow p$	The negation of a negation of a proposition is the proposition.

3	Law of the Excluded Middle	$(p \vee \neg p)$	Either something is true or it is not. This is similar to argument 1.
4	Definition of Commutation	$(p * q) \Longleftrightarrow (q * p)$	This is true when you replace $*$ with either $\wedge$ or $\vee$.
5	Definition of Associativity	$(p * q) * r \Longleftrightarrow p * (q * r)$	This is true when you replace $*$ with either $\wedge$ or $\vee$.
6	Law of the Contrapositive	$(p \Longrightarrow q) \Longleftrightarrow (\neg q \Longrightarrow \neg p)$	This is the basis for proof by contrapositive.
7	DeMorgan's Laws	$\neg (p * q) \Longleftrightarrow (\neg p \circ \neg q)$	This is true when you replace $*$ with either $\wedge$ or $\vee$ and $\circ$ with either $\vee$ or $\wedge$, respectively.
8	Definition of a Distribution	$p * (q \circ r) \Longleftrightarrow (p * q) \circ (p * r)$	This is true when you replace $*$ with either $\wedge$ or $\vee$ and $\circ$ with either $\vee$ or $\wedge$, respectively.

Proof: Here is an example of a proof by truth table, we will prove Argument 1, Contradiction.

p	$\neg p$	$(p \wedge \neg p)$	Contradiction
T	F	F	F
F	T	F	F

thus $(p \wedge \neg p) \sim$ Contradiction, which proves argument 1. QED.

Here is an example of how to discover a proof. We will prove Argument 2, Double Negative. We need to show that the double negative is equivalent to the initial proposition.

1. We start by stating that the negation of a proposition always has the opposite truth value of a proposition, thus we can write

$$q = \neg p.$$

2. The negation of q will then have the opposite truth value from q, we can write,

$$r = \neg q.$$

3. Since a proposition is either true or false, when a negation is false the starting proposition is true.
4. When r is false, then q must be true, this also means that p is false.
5. Similarly when r is true q is false, and thus p is true.
6. Therefore we see that r and p are the same.
7. Since r is the double negative of p, then we can say that the double negative of any proposition is the same as the proposition. This has been a proof by cases. QED.

Arguments involving limits

In [5] we explored the idea of a limit. We begin with the formal definition of the limit:

Definition 1 The Limit: The limit of some function $f(x)$ as x approaches some specific value a is symbolized by

$$\lim_{x \to a} f(x) = L.$$

so long as we make $f(x)$ get as close to L as we want such that x is sufficiently close to a and so long as x never really becomes a.

While this definition is adequate, it will eventually be replaced by a more accurate one.

Argument	Name	Formula	Explanation
9	Constant Multiple Rule for Limits	$\lim_{x\to a} [c\, f(x)] = c \lim_{x\to a} f(x)$	The limit of a constant multiple of a function is the constant multiple of the limit.
10	Sum and Difference Rule for Limits	$\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$	The limit of a sum is the sum of the limits.
11	Product Rule for Limits	$\lim_{x\to a} [f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$	The limit of a product is the product of the limits.

12	Quotient Rule for Limits	$\lim_{x\to a}[f(x)/g(x)] = (\lim_{x\to a} f(x))/(\lim_{x\to a} g(x))$	The limit of a quotient is the quotient of the limits.
13	Power Rule for Limits	$\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$	The limit of a power is the power of the limit.
14	Root Rule for Limits	$\lim_{x\to a}(f(x))$^ $(1/n) = (\lim_{x\to a} f(x))^{1/n}$	The limit of an nth root is the n th root of the limit.
15	Constant Limit Rule	$\lim_{x\to a}(c) = c$	The limit of a constant is that constant.
16	Limiting Value	$\lim_{x\to a}(x) = a$	
17	Power of a Limiting Value	$\lim_{x\to a}(x^n) = a^n$	
18	Limit of a Polynomial $p(x)$	$\lim_{x\to a} p(x) = p(a)$	
19	Limit Theorem 1	$f(x) \le g(x) \Rightarrow \lim_{x\to c} f(x) \le \lim_{x\to c} g(x)$	For all x on an interval $[a, b]$ where $a \le c \le b$.
20	Squeeze Theorem	If $f(x) \le h(x) \le g(x)$, then $\lim_{x\to c} h(x) = L$	This also requires $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = L$, and that $a \le c \le b$.

21	Infinite Limit	$\lim_{x \to a} f(x) = \infty$	This is written if and only if we can make $f(x)$ arbitrarily large for all values of x sufficiently close to a so long as $x \neq a$.
22	Negative Infinite Limit	$\lim_{x \to a} f(x) = -\infty$	This is written if and only if we can make $f(x)$ arbitrarily large and negative for all values of x sufficiently close to a so long as $x \neq a$.
23	Limits at Infinity	$\lim_{x \to \pm\infty} c / x^r = 0$	Infinity and zero can be thought of as inverses.
24	Infinite Polynomial Limit	$\lim_{x \to \pm\infty} p(x) = \lim_{x \to \pm\infty} a_n x^n$	The infinite limit of a polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$ is the same as the infinite limit of the highest order term of the polynomial.
25	Continuous Functions	$\lim_{x \to a} f(x) = f(a)$	A function is continuous, or smooth, if at any point a the limit of the function is the limit at that point.
26	Intermediate Value Theorem (IVT)	If $f(x)$ is continuous on an interval $[a, b]$, and if n is a number such that $f(a) \leq n \leq f(b)$, then there exists some number c such that, $a < c < b$, and $f(c) = n$.	This is a special way of saying that every continuous function will take on all values between $f(a)$ and $f(b)$.

Here is a proof of Argument 15, The Constant Multiple Rule for Limits. For this we immediately require a more precise definition of the limit than the one we have above. We need to define the absolute value,

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}. \tag{1}$$

So, we redefine the limit,

Definition 1a: The Limit. For some function $f(x)$ *then*

$$\lim_{x \to a} f(x) = L.$$

if $(\forall\ \epsilon)\ (\epsilon > 0)\ (\exists\ \delta)\ (\delta > 0)$

$$|f(x) - L| < \epsilon$$

whenever

$$0 < |x - a| < \delta.$$

Proof of Argument 15: To accomplish this proof we need to show that the limit of the constant multiple of an arbitrary function is the same as the constant multiple of the limit of the function. It seems the most straightforward way to do this is to compute both expressions and show they are equivalent.

1. Let us begin with the hypothesis, $\lim_{x \to a} [c\ f(x)]$.
2. By the definition we have some value $\epsilon > 0$ such that

   $$|\ c\ f\ (x)\ -\ L\ |\ <\ \epsilon \tag{2}$$

   whenever there is a value $\delta > 0$ such that,

   $$0 < |\ x - a\ | < \delta.$$

3. If we think about this for a while we realize that in this situation the limit L is the product of a different limit M and the constant c, by the definition of the limit.
4. This gives us a nice clue as to how to complete the proof; we need to show that (2) is equivalent to $|\ c\ |\ \ |\ f(x)\ -\ L\ |\ <\ \epsilon$.
5. We begin by rewriting (2)

   $$|\ c\ f(x)\ -\ c\,M\ |\ <\ \epsilon.$$

6. By argument 35, (the distributive property) we can rewrite this,

   $$|\ c\ |\ \ |\ f(x)\ -\ M\ |\ <\ \epsilon. \tag{3}$$

7. So, by (1) we have $\lim_{x \to a} [c\ f(x)] = M$, and by (3) we have $c \lim_{x \to a} f(x) = M$.

Thus we have $\lim_{x \to a} [c\ f(x)] = c \lim_{x \to a} f(x)$, QED.

Arguments involving differentiation

In [5] I introduced the idea of differential calculus. We will present the following definition of the derivative of a function:

Definition 2 The Derivative: The derivative of a function, $f(t)$ is given as,

$$\frac{dx}{dt} = \frac{d\,x}{d\,t} = f'(t) = D_t\,x = \lim_{\Delta t \to 0} \frac{\Delta f(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}(f(t + \Delta t) - f(t)).$$

Argument	Name	Formula	Explanation
27	Differentiability		A function is differentiable at some point a if $f'(a)$ exists.
28	Differentiability on an Interval		A function is differentiable on an interval $[a, b]$ if $f'(t)$ exists for every point $a \le t \le b$.
29	Continuity		If $f(t)$ is differentiable at $t = a$, then $f(t)$ is continuous at $t = a$.
30	Slope of a Tangent Line		The slope of a line tangent to a point a on $f(t)$ is $f'(a)$.
31	Constant Derivative Rule	$\frac{d}{dt} c = 0$	The derivative of a constant is 0.
32	Constant Multiple Rule	$\frac{d}{dt} c\ f(t) = c\,d / d\,t\ f(t)$	The derivative of a constant multiple is the constant multiple of the derivative.
33	Sum Rule	$[f(t) \pm g(t)]' = f'(t) \pm g'(t)$	The derivative of a sum is the sum of the derivatives.
34	Power Rule	$\frac{d}{dt} t^n = n\,t^{n-1}$	
35	Product Rule	$[f(t) \cdot g(t)]' = f'(t)\,g(t) + g'(t)\,f(t)$	

36	Quotient Rule	$\frac{d}{dt}[f(t)/g(t)] = (f'(t)\,g(t) - g'(t)\,f(t))/g(t)^2$	
37	Chain Rule	$\frac{dx}{dy} = \frac{dx}{dt}\,dt/dy$	This allows you to change variables in differentiation. It is a direct application of argument 11, above.

I will now prove Argument 32, The Constant Multiple Rule. We begin with the hypothesis.

1. Assume that we have $\frac{d}{dt} c\, f(t)$.
2. By Definition 2, this gives us, $\frac{d}{dt} c\, f(t) = \lim_{\Delta t \to 0} \frac{c\, f(t+\Delta t) - c\, f(t)}{\Delta t}$.
3. Factoring this we have, $\frac{d}{dt} c\, f(t) = \lim_{\Delta t \to 0} \frac{c\,[f(t+\Delta t) - f(t)]}{\Delta t}$.
4. Then by Argument 36, The Constant Multiple Rule for Limits, we now have, $\frac{d}{dt} c\, f(t) = c \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$.
5. This is equivalent, by the definition of the derivative, to $c \frac{d}{dt} f(t)$, the conclusion.
6. Thus $\frac{d}{dt} c\, f(t) = c \frac{d}{dt} f(t)$, QED.

Arguments involving integration

In [5] I introduced the idea of an integral. Given any function $f(t)$, its *antiderivative* is the function $F(t)$ such that,

$$F'(t) = f(t).$$

The most general antiderivative is called the indefinite integral and is written,

$$\int f(t)\,dt = F(t) + c.$$

Here are some arguments involving integration.

Argument	Name	Formula	Explanation
38	Constant Multiple Rule	$\int k\, f(t)\,dt = k \int f(t)\,dt$	This allows us to factor any multiplicative constants out of the integrand.

39	Sum Rule for Integrals	$\int f(t) \pm g(t)\, dt = \int f(t)\, dt \pm \int g(t)\, dt$	The integral of the sum is the sum of the integrals.
40	Power Rule for Integrals	$\int t^n\, dt = t^{n+1} / (n + 1) + c$	Here $n \neq -1$.
41	Constant Rule for Integrals	$\int k\, dt = k t + c$	
42	Substitution Rule	$\int f(g(t))\, g'(t)\, dt = \int f(v)\, dv$	Here we understand that $v = g(t)$. This is an application of the Chain Rule, (Argument 37) above.
43	Fundamental Theorem of Calculus	$\int_a^b f(t)\, dt = \int f(b)\, dt - \int f(a)\, dt$	In essence this defines an integral over an interval from a to b. Such an integral is called a *definite integral*. The values a and b are called the *limits of integration*.
44	Interchanging the Limits of ntegration	$\int_a^b f(t)\, dt = -\int_b^a f(t)\, dt$	We can interchange the limits of integration by changing the sign.
45	The Same Limits	$\int_a^a f(t)\, dt = 0$	
46	Splitting the Limits of ntegration	$\int_a^b f(t)\, dt = \int_a^c f(t)\, dt + \int_c^b f(t)\, dt$	We can split the limits of integration so long as $a \leq c \leq b$.
47	Equivalent Integrals	$\int_a^b f(t)\, dt = \int_a^b f(x)\, dx$	

48	Constant Rule for Definite Integrals	$\int_a^b c\, dt = c \cdot (b - a)$	

I will prove Argument 38, the Constant Multiple Rule for Integration. Here we begin with the definition of the integral.

1. By the definition of an integral $\int f(t)\, dt = F(t) + c$.
2. This implies $[F(t) + c]' = f(t)$.
3. By argument 32 we can multiply this by an constant k, $k\,[F(t) + c]' = k\, f(t)$.
4. We can apply this to step 1 and get, $\int k\, f(t)\, dt = k\,F(t) + k\,c$.
5. By argument xx, The Distributive Property, the right hand side of this becomes, $k\,F(t) + k\,c = k\,[F(t) + c]$.
6. By the definition of integration this is equivalent to $k \int f(t)\, dt = k[F(t) + c]$.
7. By step 5 then $\int k\, f(t)\, dt = k \int f(t)\, dt$. QED.

Arguments involving sequences and series

In [5] I introduced the idea of a series. Here I formalize that. I will make three definitions.

Definition 3 Sequences: A sequence is a list, in a specific order, of n terms. A sequence is called infinite if $n \to \infty$. A sequence may be writeen,

$$\{s_n\} = \{s_1,\ s_2,\ s_3,\ \ldots,\ s_n\}.$$

Definition 4 Limit of a Sequence: An infinite sequence $\{s(n)\}$ has a limit

$$\lim_{n \to \infty} s_n = c$$

if, $\forall\ p > 0,\ \exists\ N$ such that $s - p < s_n < c + p$ for $n > N$.

Definition 5 Series: A series is the sum of the first n terms of a sequence.

$$\sum_{i=o}^{n} a_i = a_0 + a_1 + a_2 + \ldots + a_n$$

If this is the sum of an infinite sequence, then it is called an infinite series.

$$\sum_{i=j}^{\infty} a_i = a_j + a_{j+1} + \ldots + a_a + \ldots$$

Definition 6 Sequence of Partial Sums: Associated with every infinite series is the sum of the first n terms of the infinite series

$$s_n = \sum_{i=o}^{n} a_i = a_0 + a_1 + a_2 + \ldots + a_n$$

This is called the sequence of partial sums $\{s_n\}$.

Argument	Name	Formula	Explanation
49	Convergent sequence		A sequence that has a limit is said to *converge*.
50	Divergent sequence		A sequence that does not converge is said to diverge.
51	Relevance of limit theorems		The arguments relating limits are applicable to the limits of sequences.
52	Bounded above	$s_n \leq B$	A sequence is bounded above if every value of the sequence is less than some value of B. B is called the *upper bound*.
53	Bounded below	$s_n \geq B$	A sequence is bounded below if every value of the sequence is more than some value of B. B is called the *lower bound*.
54	Monotone nondecreasing	$s_n \leq s_{n+1}$	Every subsequent term of the sequence is less than, or equal to, the previous term.
55	Monotone strictly increasing	$s_n < s_{n+1}$	Every subsequent term of the sequence is less than the previous term.
56	Monotone nonincreasing	$s_n \geq s_{n+1}$	Every subsequent term of the sequence is greater than, or equal to, the previous term.

57	Monotone strictly decreasing	$s_n > s_{n+1}$	Every subsequent term of the sequence is greater than the previous term.
58	Convergence of monotonic sequences	$\lim_{n\to\infty} s_n = c \leq B$ $\lim_{n\to\infty} s_n = c \geq B$	An infinite sequence that is bounded from above and is mononotic nondecreasing is convergent. Likewise an infinite sequence that is bounded from above and is monotonic from below is convergent.
59	Least upper bound axiom		If a set of numbers is bounded above, then there is a least upper bound $\tilde{B}$ such that all other upper bounds are greater than or equal to $\tilde{B}$.
60	Greatest lower bound theorem		If a set of numbers is bounded below, then there is a greatest lower bound $\tilde{B}$ such that all other lower bounds are less than or equal to $\tilde{B}$.
61	Proper divergence	$s_n \to \pm\infty$	A sequence that tends to infinity or negative infinity is divergent.
62	Oscillating sequences		A sequences that diverges, but is not properly divergent, is called oscillatory.
63	Monotonic sequences and convergence / divergence		A monotonic sequence either converges or is proper divergent.
64	Subsequence		If we can define a new infinite sequence from a given sequence by ignoring some terms we have a *subsequence*.

65	Limit of a subsequence		If $\{s_n\}$ is a sequence with a limit c (or $\pm\infty$), then any subsequence of $\{s_n\}$ will have a limit of c (or $\pm\infty$).
66	Oscillatory subsequences		A sequence with two subsequences that have different limits is oscillatory.
67	Limit of a subsequence of a monotonic sequence		If a subsequence of the monotonic sequence $\{s_n\}$ has a limit c (or $\pm\infty$) then $\{s_n\}$ also has a limit of c (or $\pm\infty$).
68	Cauchy condition	Given $\{s_n\}$, $\forall\, \epsilon > 0$, $\exists\, N$, $\lvert s_n - s_m \rvert < \epsilon$ when $m,\ n > N$.	Far out in the sequence, all terms are close together.
69	Cauchy criterion		If a sequence converges then it satisfied the Cauchy condition, also if the sequence satisfies the Cauchy condition it is convergent.
70	Convergence of infinite series		An infinite series converges if its sequence of partial sums is bounded.
71	Proper divergence of an infinite series		An infinite series is proper divergent if its sequence of partial sums is unbounded.
72	Oscillating infinite series		An infinite series is oscillating if its sequence of partial sums is oscillatory.

73	Zero terms		Adding or removing zero terms to a series has no effect on the series.
74	Replacement of k terms in an infinite series	$b_1 + b_2 + \dots + b_k + a_{k+1} + a_{k+2} + \dots + a_n = a_1 + a_2 + \dots + a_n + [(b_1 + b_2 + \dots + b_k) - (a_1 + a_2 + \dots + a_k)] = s_n + d$	The effect of replacing terms in an infinite series is to add a constant to the nth partial sum of the original series. In general this leaves the convergence of the original series unchanged.
75		$s_n = a_1 + a_2 + \dots + a_n$, $t_n = a_{n+1} + a_{n+2} + \dots = \sum_{j=1}^{\infty} a_{n+j} \Rightarrow S = s_n + t_n$	Given that an infinite series converges and has the sum S, then adding a series to it creates another series that converges with the sum $s_n + t_n$.

76		For a convergent series $\sum_{i=1}^{\infty} a_i$ with sum k and we have a monotinic strictly increasing sequence $\{b_i\}$ of positive integers. Now we also have $c_1 = a_1 + a_2 + \dots + a_{b_1}$ $c_2 = a_{t_1+1} + a_{t_1+2} + \dots + a_{t_2}$ $\vdots$ $c_n = a_{t_{n-1}+1} + a_{t_{n-1}+2} + \dots + a_{t_n}$ then the series $\sum_{n=1}^{\infty} c_n$ is convergent and has sum k.	The converse of this is not true. We can also say that if, after parenthesis are inserted into a given series, the new series diverges, then the original series diverges.
77	Multiplication of a series by a constant	$A = \sum_{i=1}^{\infty} a_i$ If $\forall\, k \,\exists\, c_k = d\, a_k$, then $c\,A = \sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} d\, a_k = d \sum_{k=1}^{\infty} a_k$.	

78	Sum of series	$A = \sum_{i=1}^{\infty} a_i$ $B = \sum_{j=1}^{\infty} b_j$ If $\forall\ k \exists\ c_k = a_k + b_k$, then $A + B = \sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} (a_k + b_k)$ $= \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$.	
79	Cauchy criterion for infinite series	$\sum_{i=1}^{\infty} a_i$ is convergent if and only if, $\forall \epsilon > 0$, $\exists\ N$, such that $\lvert a_{n+1} + a_{n+2} + \ldots + a_m \rvert < \epsilon$ when $m > n > N$	Thi is Argument 69 rewritten for infinite series.
80	Dominated Series		A series $\sum_{i=j}^{\infty} a_i$ with real or complex terms is dominated by the series $\sum_{i=j}^{\infty} b_i$ with nonnegative real terms so long as $\lvert a_i \rvert \leq b_i\ \forall\ i \geq j$.

81			A series $\sum_{i=1}^{\infty} a_i$ with nonegative terms and is dominated by the convergent series, $\sum_{i=1}^{\infty} b_i$ that has the sum B, is also convergent and has the sum $A \leq B$. Also, $\sum_{i=1}^{n} a_i \leq A \leq \sum_{i=1}^{n} a_i + \sum_{j=i+1}^{\infty} b_j$.

Things to do for Day Three

Continue building your library of references. Definitely begin with references [3] and [4].

Practice Problems from Day One

- The first problem was to write out three functions of t whose properties you understand. For this, I will choose:

$$f(t) = t^3 - t^2,$$

$$g(t) = (4 + t)^3,$$

and

$$h(t) = \frac{x\,t}{t^3}.$$

- The second problem was to write out each of the functions as a divided difference as in (6) in [1]. Recall that (6) in [1] is,

$$\langle x \rangle = \frac{\Delta f(t)}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

thus the three functions become,

$$f(t) = t^3 - t^2 \Rightarrow \frac{f(t + \Delta t) - f(t)}{\Delta t} = \frac{(t + \Delta t)^3 - (t + \Delta t)^2 - t^3 + t^2}{\Delta t}$$

$$=$$

$$\frac{1}{\Delta t}\left(t^3+3t^2\,\Delta t+3t\,\Delta t^2+\Delta t^3-t^2-2t\,\Delta t-\Delta t^2-t^3+t^2\right)$$

$$=\frac{1}{\Delta t}\left(3t^2\,\Delta t+3t\,\Delta t^2+\Delta t^3-2t\,\Delta t-\Delta t^2\right)$$

$$=3t^2+3t\,\Delta t+\Delta t^2-2t-\Delta t.$$

$$g(t)=(4+t)^3\Rightarrow\frac{g(t+\Delta t)-g(t)}{\Delta t}=\frac{(4+t+\Delta t)^3-(4+t)^3}{\Delta t}$$

$$g(t)=(4+t)^3\Rightarrow\frac{g(t+\Delta t)-g(t)}{\Delta t}=\frac{(4+t+\Delta t)^3-(4+t)^3}{\Delta t}$$

$$=$$

$$\frac{1}{\Delta t}\left(t^3+3t^2\,\Delta t+12t^2+3t\,\Delta t^2+48t+24t\,\Delta t+48\,\Delta t+\right.$$
$$\left.12\,\Delta t^2+\Delta t^3+64-t^3-12t^2-48t-64\right)$$

$$=$$

$$\frac{1}{\Delta t}\left(3t^2\,\Delta t+3t\,\Delta t^2+24t\,\Delta t+48\,\Delta t+12\,\Delta t^2+\Delta t^3\right)$$

$$=3t^2+3t\,\Delta t+24t+48+12\,\Delta t+\Delta t^2.$$

and,

$$h(t)=\frac{x\,t}{t^3}=(x\,t)\,t^{-3}=x\,t^{-2}\Rightarrow\frac{h(t+\Delta t)-h(t)}{\Delta t}=\frac{x\,(t+\Delta t)^{-2}-x\,t^{-2}}{\Delta t}$$

$$\Rightarrow\;=\frac{x\left[(t+\Delta t)^{-2}-t^{-2}\right]}{\Delta t}$$

$$\Rightarrow\;=\frac{x\big/(t+\Delta t)^2-x\big/t^2}{\Delta t}$$

$$\Rightarrow\;=\frac{\left[t^2\,x-x(t+\Delta t)^2\right]\big/t^2(t+\Delta t)^2}{\Delta t}$$

$$\Rightarrow\;=$$

$$\frac{1}{\Delta t}\left[t^2 x - x\left(t^2 + 2t\Delta t + \Delta t^2\right)\right] \Big/ t^2(t+\Delta t)^2$$

$$\Rightarrow \ = \frac{x\left(2t\Delta t + \Delta t^2\right) \Big/ t^2(t+\Delta t)^2}{\Delta t}$$

$$\Rightarrow \ = \frac{x(2t + \Delta t)}{t^2(t+\Delta t)^2}.$$

- The third problem is to take the derivative of each function in t. Recall that the derivative is,

$$f'(t) = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

So,

$$f('t) = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} = \lim_{\Delta t \to 0} 3t^2 + 3t\Delta t + \Delta t^2 - 2t - \Delta t$$

$$= 3t^2 + 3t(0) + (0)^2 - 2t - (0)$$

$$= 3t^2 - 2t.$$

$$g('t) = \lim_{\Delta t \to 0} \frac{g(t + \Delta t) - g(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} 3\ t^2 + 3t\Delta t + 24t + 48 + 12\Delta t + \Delta t^2$$

$$= 3\ t^2 + 3t(0) + 24t + 48 + 12(0) + (0)^2$$

$$= 3\ t^2 + 24t + 48.$$

and

$$h('t) = \lim_{\Delta t \to 0} \frac{h(t + \Delta t) - h(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{x(2t + \Delta t)}{t^2(t+\Delta t)^2}$$

$$= \frac{x(2t + 0)}{t^2(t+0)^2}$$

$$= \frac{x(2\,t\,)}{t^2(t)^2}$$

$$= \frac{2\;x\,t}{t^4}$$

$$= \frac{2\;x}{t^3}.$$

Practice Problems

Choose one of the arguments by logic and prove it is true. A good project is to prove them all.

Conclusions

I have presented a fairly good reference for beginning to explore the mathematics used in physics. This is a good beginning.

References

[1] George E. Hrabovsky, (2009), **Day 1: Introduction to Theoretical Physics**. MASTers Notes, Issue 1, (http://www.madscitech.org/notes.html).

[2] Steven Galovich, (1989), **Introduction to Mathematical Structures**, Harcourt Brace Jovanovich. This truly wonderful book is out of print, but you can still find it here and there. It is my favorite book on logic, proofs, and set theory. I picked up my copy at a used book store for only $8.

[3] Joseph Fields, (?), **A Gentle Introduction to the Art of Mathematics**. This free textbook is available from the author's website:
http://www.southernct.edu/~fields/GIAM/GIAM.pdf . This book goes much deeper than we intend to do.

[4] Michael A. Henning, (?), **An Introduction to Logic and Proof Techniques**. This is a free download from a course website:
http://www.southernct.edu/~fields/GIAM/GIAM.pdf this set of notes is at about the right level for our purposes.

[5] Martin V. Day, (2009), **An Introduction to Proofs and the Mathematical Vernacular**. This is a free download from the website:

http://www.math.vt.edu/people/day/ProofsBook. This book assumes that you have studied calculus.

[6] Dave Witte Morris, Joy Morris, (2009), **Proofs and Concepts**. This free book can be downloaded from the website: http://people.uleth.ca/~dave.morris/books/proofs+concepts.html. Sections I and II cover the topic of this writing.

Appendix: Arguments from Basic Mathematics

I am assuming that you will not need explanation for these. I include them for completeness and future reference.

Arguments by algebraic manipulation

In this section I will assume that you are familiar with the basic operations of arithmetic: addition, subtraction, multiplication, division, exponentiation, and root-taking. I will introduce a number of technical terms that will be understood without fomal definition.

- In algebra, the set of objects that you can choose variables from will be a set of numbers.
- The first set of numbers is the *natural numbers*, the counting numbers, and is symbolized $\mathbb{N}$.
- The second set of numbers are the *whole numbers*, the natural numbers and zero, denoted $\mathbb{W}$.
- The third set of numbers is the *integers*, the whole numbers and the negative of the natural numbers, denoted $\mathbb{Z}$.
- The fourth set of numbers is the *rational numbers*, fractions whose denominator and numerator are both integers, denoted $\mathbb{Q}$.
- The fifth set of numbers is the *real numbers*, the rational numbers and all irrational numbers, denoted $\mathbb{R}$.
- The sixth set of numbers is the *imaginary numbers*, these are multiples of $\mathbb{i} = \sqrt{-1}$.
- The seventh set of numbers is the *complex numbers*, the set of all real and imaginary numbers of the form $z = x + \mathbb{i}\, y$, where x and y are real. This set is denoted $\mathbb{C}$.
- A predicate in algebra is frequently called an *expression*.
- The *value* of a variable is provided when a variable is replaced with a specific choice of object.
- Finding the value of an expression is called *evaluation* of the expression. You

must be careful to be consistent in your choise of values for the variables, that is the same variable in multiple terms must all have the same values.

- When you evaluate an expression you begin by evaluating whatever terms are within grouping symbols (parentheses, brackets, etc.). Then you evaluate all exponents. Then you evaluate all products and quotients. And then you evaluate all sums and differences.
- A *polynomial* expression is an expression whose *terms* are integer powers of the variables. The highest integer power of the polynomial is called the *degree* of the polynonial. Thus, $x^2 + x + 1$ is an example of a polynomial of second degree with three terms.
- A polynomial of only one term is called a *monomial*.
- Any constant factor of a term in a polynomial is called a *coefficient*. For example, $3\,x^2$ has a coefficient of 3.
- Like terms in an expression can be combined by adding, or subtracting, coefficients as necessary. Thus, $3\,x^2 - 5\,x^2 = (3-5)\,x^2 = -2\,x^2$.
- A polynomial of degree 1 is called linear.
- A polynomial of degree 2 is called quadratic.
- A polynomial of degree 3 is called cubic.
- A polynomial of degree 4 is called quartic.
- A polynomial of degree 5 is called quintic.
- A *rational expression* is the quotient of two polynomials.
- An *equation* is an expression that states that two or more terms, or combinations of terms, have the same value. These will have an equal sign relating the relavant values, =.
  The *solution* of an equation is what you get when you evaluate that equation. A polynomial equation of a given degree will be called by the name of the polynomial of the same degree, (linear, quadratic, cubic, quartic, and quintic).
- There are five relationships between values that are not equality: two values can be greater than >, less than <, greater than or equal to ≥, less than or equal to ≤, and not equal to ≠. Any expression involving these are called *inequalities*. A polynomial inequality of a given degree will be called by the name of the polynomial of the same degree, (linear, quadratic, cubic, quartic, and quintic).

Argument	Name	Formula	Explanation
82	Fraction Multiplicaton	$\frac{a}{b} \cdot \frac{c}{d} = (a \cdot c) / (b \cdot d)$	Multiplying fractions involves first multiplying the denominators and then the numerators.

83	Fraction Division	$\frac{a}{b} \div \frac{c}{d} = \frac{a \cdot d}{b \cdot c}$	Dividing fractions is just mutiplying the dividend by the inversion of the divisor.
84	Cancellation	$\frac{a}{a} = 1$	Common factors in the denominator and numerator can be replaced by 1, term – by – term. Thus, $\frac{a+1}{a} = 1 + \frac{1}{a}$ and not 1.
85	Adding and Subtracting Fractions	$\frac{a}{b} \pm \frac{c}{d} = (a \cdot d \pm b \cdot c) / (b \cdot d)$	
86	Adding Terms	$n \cdot a + m \cdot a + b = (n + m) \cdot a + b$	We can add like terms be adding their coefficients, unlike terms cannot be added.
87	Adding Opposite Signed Integers	$a + (-b) = a - b$	Subtraction and adding opposite sign integers are equal.
88	Adding Same Signed Integers	$(-a) + (-b) = -(a + b)$	
89	Multiplying Opposite Signed Integers	$a \cdot (-b) = -(a \cdot b)$	
90	Multiplying Opposite Signed Integers	$a \cdot b = (-a) \cdot (-b)$	
91	Double Negative	$-(-a) = a$	

92	Multplying a Term by 1	$a \cdot \frac{b}{b} = a \cdot 1 = a$	We can always multiply a term by 1.
93	Multiplying Exponents	$a^m \cdot a^n = a^{m+n}$	
94	Dividing Exponents	$\frac{a^m}{a^n} = a^{m-n}$	
95	Reciprocal	$a^{-1} = \frac{1}{a}$	
96	Negative Power	$a^{-n} = 1 / a^n$	
97	Power of a Product	$(a \cdot b)^n = a^n \cdot b^n$	
98	Power of a Quotient	$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$	
99	Rewriting Division	$\frac{a}{b} = a \cdot b^{-1}$	
100	Multiplying Same Signed Integers	$a \cdot b = (-a) \cdot (-b)$	
101	Product of Roots	$(a \cdot b)$^$(1 / n) = a$^$(1 / n) \cdot b$^$(1 / n)$	
102	Quotient of Roots	(a / b)^$(1 / n) = (a$^$(1 / n)) / (b$^$(1 / n))$	
103	Power of a Root	$a^m \wedge (1 / n) = (a \wedge (1 / n))^m$	This also means that if $m = n$ then we have $\sqrt[n]{a^n} = \| a \|$.
104	Root as a Fractional Power	a^$(1 / n) = a^{\frac{1}{n}}$	
105	The General Commutative Property	$a * b = b * a$	Here we replace $*$ by either $+$ or $\cdot$.

106	The General Associative Property	$(a * b) * c = a * (b * c)$	Here we replace $*$ by either $+$ or $\cdot$.
107	Distributive Property	$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$	This is also the basis for factoring, in which case we reverse the process.
108	Multiplying Binomials	$(a + b) \cdot (c + d) = (a \cdot c) + (a \cdot d) + (b \cdot c) + (b \cdot d)$	

Proof of argument 82, Fraction Multiplication.

1. Assume that we have rational numbers $x = a / b$ and $y = c / d$.

2. By the definition of a rational number, t / s, where s and t are integers, $x \cdot y = \frac{a}{b} \cdot \frac{c}{d}$.

3. This is equivalent to the expression for integers $(a \div b) \cdot (c \div d)$.

4. We can rewrite this $\left(a \cdot \frac{1}{b}\right) \cdot \left(c \cdot \frac{1}{d}\right)$.

5. By the associative property of multiplication we can write this, $(a \cdot c) \cdot \left(\frac{1}{b} \cdot \frac{1}{d}\right)$.

6. By the definition of division we can rewrite this, $(a \cdot c) \div (b \cdot d)$.

7. By the definition of rational numbers we have, $\frac{a \cdot c}{b \cdot d}$.

8. Thus, $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$, QED.

This has been a direct proof.

Algebraic Manipulation in *Mathematica*

In addition to logical operations. *Mathematica* is good at algebraic manipulations, too.

Operation	*Mathematica* Command	Explanation
Simplify	Simplify[expr]	Performs a sequence of symbolic transformations on expr and outputs the simplest form it can find.

FullSimplify	FullSimplify[expr]	Performs an extensive sequence of symbolic transformations on expr and outputs the simplest form it can find.
Expand	Expand[expr]	Expands all products and integer powers for expr.
Factor	Factor[polynomial]	Factors a polynomial over the set of integers.
Collect	Collect[expr, pat]	Collects terms of expr that match pat.
Together	Together[rational]	Places the terms of a rational expression over a common denominator and then cancels and factors in the result.
Apart	Apart[rational]	Splits up a rational expression as a sum of terms having minimal denominators.
Cancel	Cancel[rational]	Cancels common factors in a rational expression.
PowerExpand	PowerExpand[expr]	Expands all products and powers for expr.

First there are a few things to mention regarding Simplify. If we write,

$$\texttt{Simplify}\left[\sqrt{x^2}\right]$$

$$\sqrt{x^2}$$

you might think something went wrong. Why doesn't *Mathematica* return the correct value of *x*? This is because *Mathematica* doesn't know what number system we want to use. If we say that we want to consider only positive values of *x* then we write,

```
Simplify[√(x²), x > 0]
```

```
x
```

or even better still, if we say that x is an element of the set of real numbers, or symbolically $x \in \mathbb{R}$,

```
Simplify[√(x²), x ∈ Reals]
```

```
Abs[x]
```

this is the correct answer, the square root of x^2 in the reals is the absolute value of x.

Most of the time Simplify is good enough. For cases involving so-called special functions it is often best to use FullSimplify.

```
Gamma[x + 1] Gamma[1 - x]
```

```
Gamma[1 - x] Gamma[1 + x]
```

```
Simplify[Gamma[x + 1] Gamma[1 - x]]
```

```
Gamma[1 - x] Gamma[1 + x]
```

```
FullSimplify[Gamma[x + 1] Gamma[1 - x]]
```

```
π x Csc[π x]
```

These are only a brief listing of the most basic capabilities of *Mathematica* in terms of algebraic manipulations. I invite you to explore the Documentation system and play with it.

Arguments relating to logarithms

Expanding on the ideas from the last section, we can define exponentiation and root-taking as inverse operations, similar to addition and subtraction. Thus we can define a root,

Definiiton 1: Root of an exponent: Given an exponent, a^n, its nth root is a. This is denoted $a = \sqrt[n]{a^n}$.

We can similary define the logarithm.

Definiiton 2: Logarithm of an exponent: Given an exponent, a^n, its base-a logarithm is n. This is denoted $\log_a a^n = n$.

Argument	Name	Formula	Explanation
109	Logarithm of a product	$\log_n(a\,b) = \log_n a + \log_n b$	The logarithm of a product is the sum of the logarithms.
110	Logarithm of a quotient	$\log_n\left(\frac{a}{b}\right) = \log_n a - \log_n b$	The logarithm of a quotient is the difference of the logarithms.
111	Logarithm of a power	$\log_x a^n = n \log_x a$	