The Steps in Mathematical and Scientific Problem Solving

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Introduction

Problem-solving is at the very heart of science and mathematics. All research is built around problems; from the seemingly mundane, such as the desire to solve an equation to complete a model of crop production, to modeling the effects of greenhouse gases, to the quest for quantum gravity. Normally, a scientist or mathematician gains experience in the technique of problem solving by doing assigned homework problems of ever increasing difficulty and complexity, culminating in individual research problems.

The purpose of this writing is to present what amounts to a check-list of problem-solving steps. These steps will be presented with limited explanation, such explanations will be made more explicit in future writings.

Scientific Problems

There are, broadly speaking, four types of problems in science; observational, mathematical, computational, and experimental. An observational problem places the scientist in the field to look at something that is happening. A mathematical problem occurs when you must calculate, derive, construct (geometrically or logically), mathematically model, or prove something. A computational problem occurs when you apply a computer to a set of data or to develop a computer model. An experimental problem places the scientist into an evironment, often a laboratory, where something can be measured while limiting as many other factors as possible to controlled values. These brief descriptions are, necessarily, oversimplified; few problems are limited to only one type; for example, it is rare to see a lab without a computer in it.

All problems have three essential elements: *givens*, *operations*, and *goals*. Givens are the data, expressions, materials, and assumptions that exist as you begin working the problem. Goals are the payoffs provided by solving the problem. Operations are the steps that transform the givens into the goals.

The Steps

These are the primary steps in problem solving. Each step has a number of questions within them. It is not necessary, or even likely, that each question will be relevant to the problem you decide to work on.

1. Can you choose a good problem to work on? Is there something that interests you? Can you provide a new take on an existing subject? Can you state the problem clearly? Do you understand a topic well enough to rediscover the significant resuilts for yourself?
2. Do you understand the problem you have chosen? What are the goals of the problem? What are the givens? Can you make a quick estimate of the goals? Do you have, or know, the necessary operations?
3. Do you know enough about your problem to begin planning? Can you find the necessary literature / web-based sources of information that you need? Have you been in contact with other experts who might be of help to you? Have you started a notebook?
4. Can you plan out how to solve the problem? Can you produce a diagram that will help you? Can you introduce notation that will help you? Can you see the progressive steps to the goals? Can you foresee checks that you might impose to make sure you are on the right track? Can you foresee blind alleys to avoid? Can you list the unwanted factors that might sidetrack you? Can you think of ways to make accidents work for you? What are the properties of the givens? What inferences can you draw from the givens? What are the properties of the operations you intend to use? What happens when you consider extreme cases? Can you use special values that are significant in some way? Can you simplify the problem by appealing to symmetry? Can you simplify the problem by choosing the units you will use? Can you apply reductio ad absurdum? Can you apply the contrapositive? Can you break the problem into subproblems? Do you know a similar problem that has a solution? Do you know a theorem, equation, or formula that might help? Can you plan an experiment or observation to settle the issue, or to measure something? Can you design some apparatus that will help? How will you collect any data? How will you analyze the data taken? How will you take samples? How will you measure what you have found? How will you account for measurement errors?
5. Implement your planned solution to the problem. Make sure to note any observations as they occur. Does your solution work? Can you prove that each step is correct? Can you analyze the data? Can you classify what you have found?
6. If you run into trouble, can you figure out what went wrong, and how to correct it? Try putting the problem aside for a few hours and then come back to it, did that help? Can you alter your goals to something similar? Can you recombine aspects of the problem to make things clearer? Can you simplify the problem by introducing auxiliary elements? Can you change your point of view? Can you succesfully change the notation of the problem? Can you work backwards from the goals to the givens? Can you assume that the problem is solved and then determine the properties that the solution must have? Can you relax the conditions on a goal, and restore it later? What happens when you

examine the possible different cases? Can you construct an analogy with fewer variables? Can you construct a more abstract problem? What happens when you hold all but one variable as fixed? Can you think of a way of partially meeting the goals of the problem? Are you biased in some way that prevents you from succeeding? Have you accounted for errors in measurement?

7. Can you check the solution? Did your solution use all available data? Does your solution conform to your initial estimate? Can you solve the problem by another means? Can you verify the result for specific cases? Can you reduce your result to known results? Can you use the result to derive known results? Can you summarize your results? What conclusions can you develop? Write your results in a paper.

Conclusion

At first, you will be using this as a check list, slowly working your way through each step one question at a time. As you gain experience, you will find that things become more automatic. Eventually, you will not need the list at all.

References

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