Numbers and Operations

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Mathematics as the study of objects, their relationships, and the structures created by their relationships

Mathematics is the abstract study of objects that are unique to what you are studying, and the relationships between those objects. In this way mathematics can be seen as a collection of properties of abstract structures.

Another adequate idea is that mathematics is a language. The use of symbols to represent relationships between abstract objects allows difficult ideas to be expressed without the ambiguity present in most languages. In this way mathematics can be seen as the pure form of ideas.

Perhaps the most pragmatic idea, and the one used by most scientists, is that mathematics is a way of getting numerical results from calculations. In this way mathematics is a tool to get results.

All of these ideas are good, and yet none of them is complete. Mathematics is a human invention, and stands alongside any other technology as one of the supreme wonders of the world. The fact that abstract ideas can be applied to the world around us seems magical at first glance; but these abstract ideas came from concrete applications involving, at the beginning of mathematics, the measurement of land areas and the exchange of currency. It is not surprising that from these humble beginnings the vast array of modern mathematical applications are good at describing the world, since that is where the abstractions the led to the creation of these applications came from in the first place. In this regard mathematics can be seen as a sequence of abstractions from specific ideas to general principles to new specific applications, and so on.

As we proceed we will examine the objects of different branches of mathematics, then their relationships, sometimes we will examine how we can use these objects and relationships as language, and sometimes we will examine how to use them to make actual calculations.

Numbers and Sets as Mathematical Objects

The first branch of mathematics we will examine is that of numbers and their relationships, this is often called *arithmetic*. At more advanced levels this is called *number theory*.

The first mathematical object we will examine is the idea of a collection of things that are connected by some idea or property. We will call such a collection of things as a *set*. The things collected in a set are called the *elements* of that set. We can symbolize, for example, that a is an element of the set A by writing that $a \in A$. When studying abstract sets, without specific applications or properties, we will label a general property of the set p and a general property that applies to the elements as $p(a)$, read, "The property p of element a."

We can count the elements of a set. The result of such a counting is called the number of elements, or the *cardinal number* of the set. We can symbolize this with C. The first likely abstraction was the ability to assign cardinal numbers without consideration of the sets that generate them. This abstraction produces the familiar counting numbers. The result is a new set that we call the *natural numbers* and is given the symbol $\mathbb{N}$. We can list the elements of $\mathbb{N}$ as $\{1, 2, 3, \ldots\}$, here ... translates to, "...and so on without limit." $\mathbb{N}$ can be seen as among the most fundamental of mathematical objects. In the next section we will examine some common relationships between natural numbers.

A set can have no elements, in which case it is called the *null* or *empty* set. Such a set has a cardinal number of 0. If we include this with the natural numbers we get a new set called the *whole numbers*, given the symbol $\mathbb{W}$ The list of elements then becomes $\{0, 1, 2, \ldots\}$.

Arithmetic Operations as Relationships Between Numbers

Now that we have the idea that what we think of as a number is a symbol describing how many elements are in an abstract set, how do we relate one number from another? First we can increase one by the other. This is nothing more than the act of *addition* that you are, no doubt, familiar with. This is an application of counting. If we have $a + b$ then we start with the cardinal number a and we count b more times.

We can also add something to itself a number of times. This is, of course, *multiplication*. If we have $a\,b = a \times b = a \cdot b$ we begin with the cardinal number a and add it to itself b times.

In a like manner, we can multiply something by itself a number of times. The number being multiplied is called the *base*, the number of multiplications is called the *exponent*, and the result is called the *power*. If we have a^b we begin with the base a and raise it to the power of the exponent b.

These then are the basic *operations* of arithmetic. They are sometimes called by their kind; addition being an arithmetic operation of the first kind, multiplication is of the second kind, and exponentiation being of the third kind.

Each of these operations have their *inverse operations*, that is they can run backwards. Of the first kind is the process of subtraction, if we have $a - b$ we begin with the cardinal number a and count backwards b times.

Similarly, we can subtract something by a second number a number of times. This is *division* and is the inverse operation corresponding to muiltiplication, or the inverse operation of the second kind. If we have $a \div b = a\,/\,b$ then we begin with the cardinal number a and see how many times we can subtract b from it.

What then is the inverse operation of the third kind, or the inverse of exponentiation? It turns out that there are two such operations. The first is to answer the question, "If we assume cardinal number a is base raised to a power b, what is the base of the result for a specific power c?" In other words assuming $c = a^b$, given c and b what is a? We call this a *surd*, or a *root*. We write this $c^{1/b} = \sqrt[b]{c} = a$.

The other inverse operation of exponentiation is to find the exponent given the base and the power. Again assuming $c = a^b$, given c and a what is b? We call this a *logarithm*. We would write this $\log_a c = b$.

Sets of Numbers as Having Properties Determined by the Operations Relating Numbers

In this section we will develop the entire number system of normal mathematics. We begin with $\mathbb{W}$ and proceed from there. The property that an operation performed on one or more members of a set always leads to another element of that set is called *closure*. A set that has this property is said to be closed under that operation. The first thing we note is that $\mathbb{W}$ is closed under addition, multiplication, and exponentiation. It is easy to see that it is not closed under subtraction. If we assume that we have the subtraction $a - b$, where b is greater than a, also written $b > a$, then the subtraction has no answer within the set $\mathbb{W}$ We immediately see that our number system cannot break down because of subtraction, there must be some way that we can subtract any two numbers and still get a number. What is the answer?

The solution is the invention of a new set of numbers, the negative numbers. We thus increase the definition of $\mathbb{W}$ to include the numbers -1, -2, -3, and so on. Written in list notation we have {..., -3, -2, -1, 0, 1, 2, 3, ...}. This set is called the set of *integers* and is written $\mathbb{Z}$ This changes the definition of subtraction so that, $a - b = a + (-b)$. We now see that $\mathbb{Z}$ is closed under addition, multiplication, and subtraction. Multiplication is a little bit interesting. Here we have $a\,b$ giving a positive result and $(-a)\,(-b)$ giving a positive result, while $(a)\,(-b)$ or $(-a)\,(b)$ gives a negative result.

If we examine the success of $\mathbb{Z}$ it makes us feel pretty good. Then we look at division and discover that $\mathbb{Z}$ is not closed under division. This is easy to see; if we have a / b where $b > a$ there is no integer that answers the expression.

To solve the division problem we must, once again, invent a new kind of number. In this case we invent a fraction with a natural number in both the denimator and the numerator. We also introduce a new notation for constructing a set, here we say that the set of rational numbers is the set of all combinations p / q such that both p and q are elements of $\mathbb{N}$. Symbolically we write $\mathbb{Q}$={ $p / q \mid p \in \mathbb{N} \wedge q \in \mathbb{N}$}. We see that $\mathbb{Q}$ is closed under division and exponentiation, specifically we note that $a^{-b} = 1 / a^b$.

This is great! Let's look at root-taking. We find that there is no element of $\mathbb{Q}$ that is the $\sqrt{2}$! This is a new disaster. We need to invent yet another system of numbers. We state that this new kind of number need not be an exact rational number. We choose to use the decimal numbers and a decimal point to approximate what these numbers will be. Let us examine $\sqrt{2}$ digit by digit. We know that $2^2 = 4$, and $1^2 = 1$ so the first digit will have to be 1. Obviously 1 is not the answer, the answer must be somewhere between 1 and 2, so let's try 1.5. $1.5^2 = 2.25$, too big; try 1.4, $1.4^2 = 1.96$, much better. The first digit is 1, then a decimal point, and the next digit is 4. We continue in this way, getting closer to the real answer with each new digit. The answer is actually a never-ending sequence of digits, our answer will have to come to an end with a finite number of digits eventaully. In this way our decimal expansion is only an approximation of the true value. Such number are called *irrational numbers*. There are several famous irrational numbers in addition to the square root of 2, we also have π and e, (Euler's number). The combination of $\mathbb{Q}$ and the irrationals forms the set of numbers called the *real numbers*, denoted $\mathbb{R}$ The set of real numbers is extremely important to all of mathematics.

So, now $\mathbb{R}$ is closed under root-taking, right? No, only the positive reals are closed under root-taking. What about the negative numbers? What is $\sqrt{-1}$? There is no real number that can answer this question. We must invent yet another form of number. We can do this by defining a symbol $\mathbb{i} = \sqrt{-1}$, we call this the *imaginary number*. We can create an entire set of imaginary numbers by multiplying $\mathbb{i}$ by any real number. If we combine real and imaginary numbers in the form of a real x and an imaginary number $\mathbb{i}\, y$ such that $z = x + \mathbb{i}\, y$, then z is a single *complex number*. It turns out that we need no further inventions to attain closure of all arithemtic operations. In other words, the set of complex numbers, denoted $\mathbb{C}$, is closed under all arithmetic operations and all inverse arithmetic operations.