Introduction to Grassmannians

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Grassmannians

I am not going to sugar-coat it. Grassmannians are very abstract objects whose exact nature I am uncertain of at this moment. I am using lots of resources. So I will begin with a definition.

Grassmannian: The parameter space of all k-dimensional planes in $\mathbb{C}^n$*, this is denoted* $Gr_k(\mathbb{C}^n)$.

This is a wonderful definition assuming that you know what a parameter space is, what k-dimensional planes are, and what $\mathbb{C}^n$ is. Okay I assume that we all know what a vector space is, and the $\mathbb{C}^n$ is an n-dimensional complex vector space, or the space of all complex n-tuples.

Affine Space, Locus, Variety, and Parameter Space

We begin with the idea of an *affine space*. If we have a non-void set of points, P, over some field, F, along with the operation of addition, +, such that there exists a finite-dimensional vector space, V, also over F, and we have some function such that $V \times P \longrightarrow P$, then P is an affine space assuming the following conditions are met:

1. $\forall\,(v, w) \in V \bigwedge \forall\, p \in P \Longrightarrow 0 + p = p \bigwedge (v + w) + p = v + (w + p)$.
2. $\forall\,(p, q) \in P\, \exists\,!\; v \in V$ s.t. $v + q = p$.
3. $\forall\, p \in P \bigwedge \forall\, v, w \in V \bigwedge p + v = p + w \Longrightarrow v = w$.

The dimension of P is the dimension of V. If we pick some point in $o \in P$ then we pick any other point $p \in P$, we can then uniquely determine the vector $v \in V$ such that $v = p - o$. If we choose a basis vector, $\hat{v}$, then every point is assigned as number, $x(p)$,

$$p = o + x(p)\,\hat{v}. \tag{1}$$

the number $x(p)$, is then considered the coordinate of p.

We next consider the definition of a *locus*. From basic geometry we know that a locus is a geometric figure composed of points that satisfy a definite condition. A line segment is a set of points on a line between two specified points. A circle is the locus of all points a specified distance from a central point.

This leads us to consider a set of polynomial equations, $f_i \in \{x_1, \ldots, x_n\}$, if this is defined on some field, $\mathbb{F}^n$,

$$V = \{p \in \mathbb{F}^n \mid f_i(p) = 0\} \subset \mathbb{F}^n \tag{2}$$

where the zeroes of the polynomials form the locus. This locus is called a *variety*. If, the field is $\mathbb{C}^n$, so that $f_i \in k[z_1, \ldots, z_n]$, a set of smooth complex curves on an affine space, then the variety is called an *affine variety*.

If we have a varierty, $\mathcal{H}$, with a mapping, φ, between the points of $\mathcal{H}$ and a collection of varieties $\{\mathcal{X}_p \subset \mathbb{P}^n\}$, where $\mathbb{P}^n$ is the n-space of polynomials. This is called a parameter space if for every variety, B the association of a family of varieties, $\mathcal{V} \subset B \times \mathbb{P}^n \in \mathcal{X}$ of $\varphi : B \to \mathcal{V}$ iduces a bijection.

References

[1] Alexander Young, "Math 595 Notes; The Grassmannian"

[2] Joe Harris, Algebraic Geometry

[3] MacLane, Birkhoff, Algebra