Relativistic Viscous Fluids

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Introduction

This talk will proceed in the following way, all of this work being done in Mathematica using xAct:

- We will present the stress-energy tensor for the perfect fluid.
- We will then show how we can get the fluid equations of motion for a perfect fluid using the principle of source conservation applied to the stress-energy tensor.
- We will present the stress-energy tensor for a viscous fluid.
- I will show the xAct implementation of this tensor.
- I will then show how we can get the fluid equations of motion for a viscous fluid using the principle of source conservation applied to the stress-energy tensor.

The Stress-Energy Tensor for a Perfect Fluid

Given the pressure p, the density ρ, the metric tensor $g_{\alpha\beta}$, and the 4-velocity u^{α}, we can write the stress energy tensor for a perfect fluid

$$T^{\alpha\beta} = p\, g_{\alpha\beta} + (\rho + p)\, u^{\alpha}\, u^{\beta}. \qquad (1)$$

How do we tell Mathematica to write this? First we have to load xAct.

In[1]:=
```
<< xAct`xTensor`;
<< xAct`xPert`;
<< xAct`xCoba`;
<< xAct`SymManipulator`;
$DefInfoQ = False;
$CVVerbose = False;
```

The Stress-Energy Tensor for a Perfect Fluid II

Then we define our spacetime manifold.

```
In[7]:= DefManifold[ℳ, 4, {α, β, γ, ι, λ, μ, ν}];
```

Then we define the abstract metric.

```
In[8]:= DefMetric[-1, g[-μ, -ν], CD, {";", "∇"}, PrintAs → "g"];
```

We now establish our chart. We will label it cb for coordinate basis

```
In[9]:= DefChart[cb, ℳ, {0, 1, 2, 3}, {t[], r[], θ[], ϕ[]}]
```

Then we can specify how we want these written in output,

```
In[10]:= cb /: CIndexForm[0, cb] := "t";

In[11]:= cb /: CIndexForm[1, cb] := "r";

In[12]:= cb /: CIndexForm[2, cb] := "θ";

In[13]:= cb /: CIndexForm[3, cb] := "ϕ";
```

The Stress-Energy Tensor for a Perfect Fluid III

We are using geometrized units in this work where $G = 1$ and $c = 1$. We need to define the constant for the central mass, M.

In[14]:=
```
DefConstantSymbol[M]
```

We will adopt the Schwarzschild metric.

In[16]:=
```
MatrixForm[met = DiagonalMatrix[{1 - (2 M) / r[], (1 - (2 M) / r[])^-1, -r[]^2, 2 r[] Sin[θ[]]^2}]]
```

Out[16]//MatrixForm=

$$\begin{pmatrix} 1 - \frac{2M}{r} & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{2M}{r}} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & 2\, r\, \text{Sin}[\theta]^2 \end{pmatrix}$$

The Stress-Energy Tensor for a Perfect Fluid IV

We then enter this into our coordinate basis.

In[17]:=
```
MetricInBasis[g, -cb, met] // TableForm
```

Out[17]//TableForm=

$g_{tt} \to 1 - \frac{2M}{r}$	$g_{tr} \to 0$	$g_{t\theta} \to 0$	$g_{t\phi} \to 0$
$g_{rt} \to 0$	$g_{rr} \to \frac{1}{1-\frac{2M}{r}}$	$g_{r\theta} \to 0$	$g_{r\phi} \to 0$
$g_{\theta t} \to 0$	$g_{\theta r} \to 0$	$g_{\theta\theta} \to -r^2$	$g_{\theta\phi} \to 0$
$g_{\phi t} \to 0$	$g_{\phi r} \to 0$	$g_{\phi\theta} \to 0$	$g_{\phi\phi} \to 2\,r\,\text{Sin}[\theta]^2$

We can show the symmetries.

In[18]:=
```
TensorValues@g
```

Out[18]= $\text{FoldedRule}[\{g_{rt} \to g_{tr}, g_{\theta t} \to g_{t\theta}, g_{\theta r} \to g_{r\theta}, g_{\phi t} \to g_{t\phi}, g_{\phi r} \to g_{r\phi}, g_{\phi\theta} \to g_{\theta\phi}\},$
$\{g_{tt} \to 1 - (2\,M)/r, g_{tr} \to 0, g_{t\theta} \to 0, g_{t\phi} \to 0, g_{rr} \to 1/(1 - (2\,M)/r),$
$g_{r\theta} \to 0, g_{r\phi} \to 0, g_{\theta\theta} \to -r^2, g_{\theta\phi} \to 0, g_{\phi\phi} \to 2\,r\,\text{Sin}[\theta]^2\}]$

The Stress-Energy Tensor for a Perfect Fluid V

We can also enter it directly using CTensor.

```
In[19]:= gm = CTensor[met, {-cb, -cb}];
```

```
In[20]:= SetCMetric[gm, -cb];
```

The $t\,t$ component of the tensor can be displayed.

```
In[21]:= gm[{0, -cb}, {0, -cb}]
```

```
Out[21]= 1 - (2 M) / r
```

We then define the density field.

```
In[23]:= DefTensor[ρ[], M];
```

Here we define the 4-velocity.

```
In[246]:= u = CTensor[{1, 0, 0, 0}, {-cb}];
```

The Stress-Energy Tensor for a Perfect Fluid VI

For the perfect fluid we need to define a pressure field.

In[25]:=
```
DefTensor[p[], M];
```

The stress-energy tensor for the perfect fluid is,

In[247]:=
```
Tf[α_, β_] := (ρ[] × u[-α] × u[-β]) - (p[] × u[-α] × u[-β]) + p[] × gm[-α, -β]
```

In[248]:=
```
Tf[α, β]
```

Out[248]=

$$\begin{pmatrix} -p+p\left(1-\frac{2M}{r}\right)+\rho & 0 & 0 & 0 \\ 0 & \frac{p}{1-\frac{2M}{r}} & 0 & 0 \\ 0 & 0 & -p\ r^2 & 0 \\ 0 & 0 & 0 & 2\ p\ r\ \mathrm{Sin}[\theta]^2 \end{pmatrix}_{\alpha\beta}$$

We can specify a particular set of components.

In[249]:=
```
Tf[{-0, cb}, {-0, cb}]
```

Out[249]= $-p+p\left(1-\frac{2M}{r}\right)+\rho$

The Stress-Energy Tensor for a Perfect Fluid VII

We can make a command to do this,

In[250]:=
```
SET[α_, β_] := Tf[{-α, cb}, {-β, cb}]
```

In[251]:=
```
SET[0, 0]
```

Out[251]= $-p + p\left(1 - \frac{2M}{r}\right) + \rho$

In[252]:=
```
SET[0, 1]
```

Out[252]= 0

In[253]:=
```
SETTable = Table[Row[{"T"Grid[{{α,β}}], "→", SET[α, β]}], {α, 0, 3}, {β, 0, 3}] // TableForm
```

Out[253]//TableForm=

$T_{0\ 0} \to -p + p\left(1 - \frac{2M}{r}\right) + \rho$	$T_{0\ 1} \to 0$	$T_{0\ 2} \to 0$	$T_{0\ 3} \to 0$
$T_{1\ 0} \to 0$	$T_{1\ 1} \to \frac{p}{1 - \frac{2M}{r}}$	$T_{1\ 2} \to 0$	$T_{1\ 3} \to 0$
$T_{2\ 0} \to 0$	$T_{2\ 1} \to 0$	$T_{2\ 2} \to -p\, r^2$	$T_{2\ 3} \to 0$
$T_{3\ 0} \to 0$	$T_{3\ 1} \to 0$	$T_{3\ 2} \to 0$	$T_{3\ 3} \to 2\, p\, r\, \mathrm{Sin}[\theta]^2$

Using the Source Conservation Law to Derive the Equations of a Perfect Fluid

The source conservation law is often called the conservation of 4-momentum. Why the distinction? There is no general conservation of 4-momentum in general relativity, only in the source of the gravitation field.

We can write this conservation law

$$T^{\alpha\beta}{}_{;\gamma} = 0. \tag{2}$$

In our case, by (1) we can write this, in terms of the velocity of the fluid v,

$$T^{00}{}_{,0} + T^{00}{}_{,j} = \partial_t \rho + \rho\, v^i{}_{,j} = 0. \tag{3}$$

Looking at this long enough, we will realize that it is the continuity equation.

We also have the spatial components

$$T^{i\,j}{}_{,0} + T^{i\,j}{}_{,j} = \partial_t \rho\, v^i + \left(\rho\, v^i\, v^j\right)_{,j} + p_{,i} = 0. \tag{4}$$

This is none other than the Euler equation for a fluid.

Using the Source Conservation Law to Derive the Equations of a Perfect Fluid II

In general we can write the Christoffel symbols

In[32]:=
```
Christoffel[CD, PDcb][α, -β, -γ]
```

Out[32]= $\Gamma[\nabla,\mathcal{D}]^{\alpha}{}_{\beta\gamma}$

We can calculate the covariant derivative of the metric.

In[232]:=
```
cdg = CovDOfMetric[gm];
```

We can calculate the Christoffel symbols.

In[233]:=
```
chr = Christoffel[cdg, PDcb];
```

We can determine a specific Christoffel symbol

In[234]:=
```
chr[{0, cb}, {0, -cb}, {1, -cb}]
```

Out[234]= $-\frac{M}{2 M r - r^2}$

We can make a command to do this,

In[235]:=
```
Γ[α_, β_, γ_] := chr[{α, cb}, {β, -cb}, {γ, -cb}]
```

In[237]:=
```
Γ[0, 0, 1]
```

Out[237]= $-\frac{M}{2 M r - r^2}$

We can get all the components.

In[37]:=
```
ChrisTable =
 Table[Row[{"Γ"^Grid[{{α}}]_Grid[{{"",β,γ}}], "→", Γ[α, β, γ]}], {α, 0, 3}, {β, 0, 3}, {γ, 0, 3}] // TableForm
```

Out[37]//TableForm=

$\Gamma^{0}{}_{0\ 0}\to 0$	$\Gamma^{0}{}_{1\ 0}\to -\frac{M}{2\,M\,r-r^{2}}$	$\Gamma^{0}{}_{2\ 0}\to 0$	$\Gamma^{0}{}_{3\ 0}\to 0$
$\Gamma^{0}{}_{0\ 1}\to -\frac{M}{2\,M\,r-r^{2}}$	$\Gamma^{0}{}_{1\ 1}\to 0$	$\Gamma^{0}{}_{2\ 1}\to 0$	$\Gamma^{0}{}_{3\ 1}\to 0$
$\Gamma^{0}{}_{0\ 2}\to 0$	$\Gamma^{0}{}_{1\ 2}\to 0$	$\Gamma^{0}{}_{2\ 2}\to 0$	$\Gamma^{0}{}_{3\ 2}\to 0$
$\Gamma^{0}{}_{0\ 3}\to 0$	$\Gamma^{0}{}_{1\ 3}\to 0$	$\Gamma^{0}{}_{2\ 3}\to 0$	$\Gamma^{0}{}_{3\ 3}\to 0$
$\Gamma^{1}{}_{0\ 0}\to \frac{M\,(2\,M-r)}{r^{3}}$	$\Gamma^{1}{}_{1\ 0}\to 0$	$\Gamma^{1}{}_{2\ 0}\to 0$	$\Gamma^{1}{}_{3\ 0}\to 0$
$\Gamma^{1}{}_{0\ 1}\to 0$	$\Gamma^{1}{}_{1\ 1}\to \frac{M}{2\,M\,r-r^{2}}$	$\Gamma^{1}{}_{2\ 1}\to 0$	$\Gamma^{1}{}_{3\ 1}\to 0$
$\Gamma^{1}{}_{0\ 2}\to 0$	$\Gamma^{1}{}_{1\ 2}\to 0$	$\Gamma^{1}{}_{2\ 2}\to -2\,M+r$	$\Gamma^{1}{}_{3\ 2}\to 0$
$\Gamma^{1}{}_{0\ 3}\to 0$	$\Gamma^{1}{}_{1\ 3}\to 0$	$\Gamma^{1}{}_{2\ 3}\to 0$	$\Gamma^{1}{}_{3\ 3}\to \frac{(2\,M-r)\,\mathrm{Sin}[\theta]^{2}}{r}$
$\Gamma^{2}{}_{0\ 0}\to 0$	$\Gamma^{2}{}_{1\ 0}\to 0$	$\Gamma^{2}{}_{2\ 0}\to 0$	$\Gamma^{2}{}_{3\ 0}\to 0$
$\Gamma^{2}{}_{0\ 1}\to 0$	$\Gamma^{2}{}_{1\ 1}\to 0$	$\Gamma^{2}{}_{2\ 1}\to \frac{1}{r}$	$\Gamma^{2}{}_{3\ 1}\to 0$
$\Gamma^{2}{}_{0\ 2}\to 0$	$\Gamma^{2}{}_{1\ 2}\to \frac{1}{r}$	$\Gamma^{2}{}_{2\ 2}\to 0$	$\Gamma^{2}{}_{3\ 2}\to 0$
$\Gamma^{2}{}_{0\ 3}\to 0$	$\Gamma^{2}{}_{1\ 3}\to 0$	$\Gamma^{2}{}_{2\ 3}\to 0$	$\Gamma^{2}{}_{3\ 3}\to \frac{\mathrm{Sin}[2\,\theta]}{r}$
$\Gamma^{3}{}_{0\ 0}\to 0$	$\Gamma^{3}{}_{1\ 0}\to 0$	$\Gamma^{3}{}_{2\ 0}\to 0$	$\Gamma^{3}{}_{3\ 0}\to 0$
$\Gamma^{3}{}_{0\ 1}\to 0$	$\Gamma^{3}{}_{1\ 1}\to 0$	$\Gamma^{3}{}_{2\ 1}\to 0$	$\Gamma^{3}{}_{3\ 1}\to \frac{1}{2\,r}$
$\Gamma^{3}{}_{0\ 2}\to 0$	$\Gamma^{3}{}_{1\ 2}\to 0$	$\Gamma^{3}{}_{2\ 2}\to 0$	$\Gamma^{3}{}_{3\ 2}\to \mathrm{Cot}[\theta]$
$\Gamma^{3}{}_{0\ 3}\to 0$	$\Gamma^{3}{}_{1\ 3}\to \frac{1}{2\,r}$	$\Gamma^{3}{}_{2\ 3}\to \mathrm{Cot}[\theta]$	$\Gamma^{3}{}_{3\ 3}\to 0$

Using the Source Conservation Law to Derive the Equations of a Perfect Fluid III

We can write the covariant derivative of the stress-energy tensor.

In[38]:=
```
cdstpf[α_, β_, γ_] := cdg[-α][Tf[β, γ]]
```

We can write this,

In[265]:=
```
cdstpf[α, β, γ] // ToCanonical // Simplify
```

Out[265]=

$$\left(\begin{array}{cccc}
\left|\begin{array}{c} -\frac{2 M (\mathcal{D}_t p)}{r}+\mathcal{D}_t \rho \\ \bullet \\ -\frac{2 M (\mathcal{D}_\theta p)}{r}+\mathcal{D}_\theta \rho \\ -\frac{2 M (\mathcal{D}_\phi p)}{r}+\mathcal{D}_\phi \rho \end{array}\right| & \left|\begin{array}{c} \frac{M(-p+\rho)}{(2 M-r) r} \\ 0 \\ 0 \\ 0 \end{array}\right| & \left|\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right| & \left|\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right| \\
\left|\begin{array}{c} \frac{M(-p+\rho)}{(2 M-r) r} \\ 0 \\ 0 \\ 0 \end{array}\right| & \left|\begin{array}{c} \frac{\mathcal{D}_t p}{1-\frac{2 M}{r}} \\ -\frac{r (\mathcal{D}_r p)}{2 M-r} \\ \frac{\mathcal{D}_\theta p}{1-\frac{2 M}{r}} \\ \frac{\mathcal{D}_\phi p}{1-\frac{2 M}{r}} \end{array}\right| & \left|\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right| & \left|\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right| \\
\left|\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right| & \left|\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right| & \left|\begin{array}{c} -r^2 (\mathcal{D}_t p) \\ -r^2 (\mathcal{D}_r p) \\ -r^2 (\mathcal{D}_\theta p) \\ -r^2 (\mathcal{D}_\phi p) \end{array}\right| & \left|\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right| \\
\left|\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right| & \left|\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right| & \left|\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right| & \left|\begin{array}{c} 2 r \,\mathrm{Sin}[\theta]^2 (\mathcal{D}_t p) \\ 2 r \,\mathrm{Sin}[\theta]^2 (\mathcal{D}_r p) \\ 2 r \,\mathrm{Sin}[\theta]^2 (\mathcal{D}_\theta p) \\ 2 r \,\mathrm{Sin}[\theta]^2 (\mathcal{D}_\phi p) \end{array}\right|
\end{array}\right)^{\beta\gamma\alpha}$$

We can specify the specific components of this. Here we have the covariant derivative of the time-time component.

In[40]:=
```
cdstpf[{0, cb}, {0, cb}, {0, cb}] // ToCanonical // Simplify
```

Out[40]= $-\left(\left(2 M (\mathcal{D}_t p)\right) / r\right) + \mathcal{D}_t \rho$

We can write a program for this,

In[41]:=
```
cdsetpf[α_, β_, γ_] := cdstpf[{α, cb}, {β, cb}, {γ, cb}]
```

In[255]:=
```
cdsetpf[0, 0, 0] // ToCanonical // Simplify
```

Out[255]= $-\frac{2\,M\,(\mathcal{D}_t\,p)}{r} + \mathcal{D}_t\,\rho$

In[256]:=
```
cdsetpf[1, 0, 0] // ToCanonical // Simplify
```

Out[256]= $\left|\begin{matrix} -\frac{2\,M\,(\mathcal{D}_t\,p)}{r}+\mathcal{D}_t\,\rho \\ -\frac{2\,M\,(\mathcal{D}_r\,p)}{r}+\mathcal{D}_r\,\rho \\ -\frac{2\,M\,(\mathcal{D}_\theta\,p)}{r}+\mathcal{D}_\theta\,\rho \\ -\frac{2\,M\,(\mathcal{D}_\phi\,p)}{r}+\mathcal{D}_\phi\,\rho \end{matrix}\right|_{-1}$

In[257]:=
```
cdsetpf[2, 0, 0] // ToCanonical // Simplify
```

Out[257]= $\left|\begin{matrix} -\frac{2\,M\,(\mathcal{D}_t\,p)}{r}+\mathcal{D}_t\,\rho \\ -\frac{2\,M\,(\mathcal{D}_r\,p)}{r}+\mathcal{D}_r\,\rho \\ -\frac{2\,M\,(\mathcal{D}_\theta\,p)}{r}+\mathcal{D}_\theta\,\rho \\ -\frac{2\,M\,(\mathcal{D}_\phi\,p)}{r}+\mathcal{D}_\phi\,\rho \end{matrix}\right|_{-2}$

In[258]:=
```
cdsetpf[3, 0, 0] // ToCanonical // Simplify
```

Out[258]= $\left|\begin{matrix} -\frac{2\,M\,(\mathcal{D}_t\,p)}{r}+\mathcal{D}_t\,\rho \\ -\frac{2\,M\,(\mathcal{D}_r\,p)}{r}+\mathcal{D}_r\,\rho \\ -\frac{2\,M\,(\mathcal{D}_\theta\,p)}{r}+\mathcal{D}_\theta\,\rho \\ -\frac{2\,M\,(\mathcal{D}_\phi\,p)}{r}+\mathcal{D}_\phi\,\rho \end{matrix}\right|_{-3}$

Using the Source Conservation Law to Derive the Equations of a Perfect Fluid IV

Putting together all of the derivatives of the time components

In[261]:=
```
tcdsetpft = Total[Table[cdsetpf[α, 0, 0], {α, 0, 3}]] // ToCanonical // Simplify
```

Out[261]=

$$\frac{1}{r}\left(\begin{vmatrix} -2\,M\,(\mathcal{D}_t\,p)+r\,(\mathcal{D}_t\,\rho) \\ -2\,M\,(\mathcal{D}_r\,p)+r\,(\mathcal{D}_r\,\rho) \\ -2\,M\,(\mathcal{D}_\theta\,p)+r\,(\mathcal{D}_\theta\,\rho) \\ -2\,M\,(\mathcal{D}_\phi\,p)+r\,(\mathcal{D}_\phi\,\rho) \end{vmatrix}_{-3} + \begin{vmatrix} -2\,M\,(\mathcal{D}_t\,p)+r\,(\mathcal{D}_t\,\rho) \\ -2\,M\,(\mathcal{D}_r\,p)+r\,(\mathcal{D}_r\,\rho) \\ -2\,M\,(\mathcal{D}_\theta\,p)+r\,(\mathcal{D}_\theta\,\rho) \\ -2\,M\,(\mathcal{D}_\phi\,p)+r\,(\mathcal{D}_\phi\,\rho) \end{vmatrix}_{-2} + \begin{vmatrix} -2\,M\,(\mathcal{D}_t\,p)+r\,(\mathcal{D}_t\,\rho) \\ -2\,M\,(\mathcal{D}_r\,p)+r\,(\mathcal{D}_r\,\rho) \\ -2\,M\,(\mathcal{D}_\theta\,p)+r\,(\mathcal{D}_\theta\,\rho) \\ -2\,M\,(\mathcal{D}_\phi\,p)+r\,(\mathcal{D}_\phi\,\rho) \end{vmatrix}_{-1} - 2\,M\,(\mathcal{D}_t\,p) + r\,(\mathcal{D}_t\,\rho)\right)$$

Setting this to 0 will give us the continuity equation for a perfect fluid in Schwarzschild spacetime. This is how (3) is adapted to our specific geometry.

Using the Source Conservation Law to Derive the Equations of a Perfect Fluid V

In order to get the equations of momentum we need to apply a projection tensor.

We begin by defining the projection tensor.

In[433]:=
```
P[α_, β_] := gm[-α, -β] + u[-α] × u[-β]
```

In[434]:=
```
P[α, β]
```

Out[434]=

$$\begin{pmatrix} 2-\frac{2M}{r} & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{2M}{r}} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & 2\, r\, \mathrm{Sin}[\theta]^2 \end{pmatrix}_{\alpha\beta}$$

We can specify a particular set of components.

In[142]:=
```
P[{-0, cb}, {-0, cb}] // Simplify
```

Out[142]=

$$2-\frac{2M}{r}$$

We can make a command to do this,

In[143]:=
```
PT[α_, β_] := P[{-α, cb}, {-β, cb}] // Simplify
```

In[144]:=
```
PT[0, 0]
```

Out[144]= $2 - \frac{2M}{r}$

Out[145]//TableForm=

$P_{0\,0} \to 2 - \frac{2M}{r}$	$P_{0\,1} \to 0$	$P_{0\,2} \to 0$	$P_{0\,3} \to 0$
$P_{1\,0} \to 0$	$P_{1\,1} \to \frac{1}{1 - \frac{2M}{r}}$	$P_{1\,2} \to 0$	$P_{1\,3} \to 0$
$P_{2\,0} \to 0$	$P_{2\,1} \to 0$	$P_{2\,2} \to -r^2$	$P_{2\,3} \to 0$
$P_{3\,0} \to 0$	$P_{3\,1} \to 0$	$P_{3\,2} \to 0$	$P_{3\,3} \to 2\,r\,\text{Sin}[\theta]^2$

Using the Source Conservation Law to Derive the Equations of a Perfect Fluid VI

So we then write,

```
In[481]:= mom[α_, β_, γ_, μ_, ν_] := P[μ, ν] × cdstpf[α, β, γ] // ToCanonical // Simplify

In[482]:= mom[α, β, γ, μ, ν]
```

Out[482]=

(●, 0, 0, 0)	(0, ●, 0, 0)	$(0,\ 0,\ -r\,(-2M(\mathcal{D}_t\,p)+r(\mathcal{D}_t\,\rho)),\ 0)$	(0, 0, 0, ●)
(●, 0, 0, 0)	(0, ●, 0, 0)	(0, 0, ●, 0)	(0, 0, 0, ●)
(●, 0, 0, 0)	(0, ●, 0, 0)	$(0,\ 0,\ -r\,(-2M(\mathcal{D}_\theta\,p)+r(\mathcal{D}_\theta\,\rho)),\ 0)$	(0, 0, 0, ●)
(●, 0, 0, 0)	(0, ●, 0, 0)	$(0,\ 0,\ -r\,(-2M(\mathcal{D}_\phi\,p)+r(\mathcal{D}_\phi\,\rho)),\ 0)$	(0, 0, 0, ●)

$\left(\frac{2M(M-r)(p-\rho)}{(2M-r)r^2},\ 0,\ 0,\ 0\right)$	$\left(0,\ \frac{M(p-\rho)}{(-2M+r)^2},\ 0,\ 0\right)$	$\left(0,\ 0,\ \frac{Mr(p-\rho)}{2M-r},\ 0\right)$	$\left(0,\ 0,\ 0,\ \frac{2M\,\mathrm{Sin}[\theta]^2(-p+\rho)}{2M-r}\right)$
(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)
(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)
(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)

$\left(\frac{2M(M-r)(p-\rho)}{(2M-r)r^2},\ 0,\ 0,\ 0\right)$	$\left(0,\ \frac{M(p-\rho)}{(-2M+r)^2},\ 0,\ 0\right)$	$\left(0,\ 0,\ \frac{Mr(p-\rho)}{2M-r},\ 0\right)$	$\left(0,\ 0,\ 0,\ \frac{2M\,\mathrm{Sin}[\theta]^2(-p+\rho)}{2M-r}\right)$
(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)
(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)
(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)

$\left(\frac{2(M-r)(\mathcal{D}_t\,p)}{2M-r},\ 0,\ 0,\ 0\right)$	$\left(0,\ \frac{r^2(\mathcal{D}_t\,p)}{(-2M+r)^2},\ 0,\ 0\right)$	$\left(0,\ 0,\ \frac{r^3(\mathcal{D}_t\,p)}{2M-r},\ 0\right)$	$\left(0,\ 0,\ 0,\ \frac{2r\,\mathrm{Sin}[\theta]^2(\mathcal{D}_t\,p)}{1-\frac{2M}{r}}\right)$
$\left(\frac{2(M-r)(\mathcal{D}_r\,p)}{2M-r},\ 0,\ 0,\ 0\right)$	$\left(0,\ \frac{r^2(\mathcal{D}_r\,p)}{(-2M+r)^2},\ 0,\ 0\right)$	$\left(0,\ 0,\ \frac{r^3(\mathcal{D}_r\,p)}{2M-r},\ 0\right)$	$\left(0,\ 0,\ 0,\ -\frac{2r^2\,\mathrm{Sin}[\theta]^2(\mathcal{D}_r\,p)}{2M-r}\right)$
$\left(\frac{2(M-r)(\mathcal{D}_\theta\,p)}{2M-r},\ 0,\ 0,\ 0\right)$	$\left(0,\ \frac{r^2(\mathcal{D}_\theta\,p)}{(-2M+r)^2},\ 0,\ 0\right)$	$\left(0,\ 0,\ \frac{r^3(\mathcal{D}_\theta\,p)}{2M-r},\ 0\right)$	$\left(0,\ 0,\ 0,\ \frac{2r\,\mathrm{Sin}[\theta]^2(\mathcal{D}_\theta\,p)}{1-\frac{2M}{r}}\right)$
$\left(\frac{2(M-r)(\mathcal{D}_\phi\,p)}{2M-r},\ 0,\ 0,\ 0\right)$	$\left(0,\ \frac{r^2(\mathcal{D}_\phi\,p)}{(-2M+r)^2},\ 0,\ 0\right)$	$\left(0,\ 0,\ \frac{r^3(\mathcal{D}_\phi\,p)}{2M-r},\ 0\right)$	$\left(0,\ 0,\ 0,\ \frac{2r\,\mathrm{Sin}[\theta]^2(\mathcal{D}_\phi\,p)}{1-\frac{2M}{r}}\right)$

(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)
(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)
(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)
(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)

(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)
(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)
(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)
(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)

$(2(M-r)\,r\,(\mathcal{D}_t\,p),\ 0,\ 0,\ 0)$
$(2(M-r)\,r\,(\mathcal{D}_r\,p),\ 0,\ 0,\ 0)$
$(2(M-r)\,r\,(\mathcal{D}_\theta\,p),\ 0,\ 0,\ 0)$
$(2(M-r)\,r\,(\mathcal{D}_\phi\,p),\ 0,\ 0,\ 0)$

(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)
(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)
(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)
(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)

(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)
(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)
(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)
(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)

The Stress-Energy Tensor for a Viscous Fluid I

It is important to realize that for a Non-Perfect Fluid the stress-energy tensor will have the form of the stress-energy tensor plus some additional terms,

$$T^{\alpha\beta} = \text{Stress} - \text{Energy Tensor}: \text{ Perfect Fluid } + \text{ Viscous Contributions } + \text{ The Generation of Energy Fluxes.} \tag{5}$$

We will adopt the Eckert frame for our presentation as this is most generally applicable (though not for some many-body collision cases).

The stress-energy tensor for the Eckert frame of a viscous fluid exhibiting heat flow, having an anisotropic stress tensor $\pi^{\alpha\beta}$, the viscous bulk pressure Π, a projection tensor $P^{\alpha\beta}$, the heat-flux 4-vector q^{α},

$$T^{\alpha\beta} = \rho\, u^\alpha\, u^\beta + \pi^{\alpha\beta} + (p + \Pi\,)\, P^{\alpha\beta} + q^\alpha\, u^\beta + q^\beta\, u^\alpha. \tag{6}$$

$$P^{\alpha\beta} = g^{\alpha\beta} + u^\alpha\, u^\beta. \tag{7}$$

$$\Pi = p - p_{\text{eq}}. \tag{8}$$

We also have the heat flux 4-vector,

$$q^\alpha = \left(0,\, q^{\hat{i}}\right) \tag{9}$$

where the hat represents spatial components in the comoving frame of the fluid.

The Stress-Energy Tensor for a Viscous Fluid II

We then establish our equilibrium pressure.

In[435]:=
```
DefConstantSymbol[peq]
```

Then we define the viscous bulk pressure.

In[436]:=
```
Π[] := p[] - peq
```

The heat flux 4 - vector is.

In[438]:=
```
q = CTensor[{0, q1, q2, q3}, {-cb}];
```

In[441]:= $c = \text{CTensor}\left[\left\{\sqrt{\frac{1}{1-\frac{2M}{r}}}, 0, 0, 0\right\}, \{-cb\}\right];$

In[442]:=
```
DefTensor[S[], M];
```

In[444]:= $\sigma[\alpha\_, \beta\_] := \sqrt{3}\, S[]\left(c[-\alpha] \times c[-\beta] - \frac{1}{3}(u[-\alpha] \times u[-\beta] - gm[-\alpha, -\beta])\right)$

In[445]:=
```
σ[α, β]
```

Out[445]=
$$\begin{pmatrix} \sqrt{3}\left(\frac{1}{1-\frac{2M}{r}} - \frac{2M}{3r}\right) S & 0 & 0 & 0 \\ 0 & \frac{S}{\sqrt{3}\left(1-\frac{2M}{r}\right)} & 0 & 0 \\ 0 & 0 & -\frac{r^2 S}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & \frac{2 r S \sin[\theta]^2}{\sqrt{3}} \end{pmatrix}_{\alpha\beta}$$

The Stress-Energy Tensor for a Viscous Fluid III

In[453]:=
```
Tvf[α_, β_] :=
  ρ[] × u[-α] × u[-β] + σ[α, β] + (p[] + Π[]) P[α, β] + q[-α] × u[-β] + q[-β] × u[-α]
```

In[455]:=
```
Tvf[α, β] // ToCanonical
```

Out[455]=

●	q1	q2	q3
q1	●	0	0
q2	0	$\text{peq } r^2 - 2\text{ p } r^2 - \frac{r^2\text{ S}}{\sqrt{3}}$	0
q3	0	0	●

$\alpha\beta$

We can specify a particular set of components.

In[456]:=
```
Tvf[{-0, cb}, {-0, cb}]
```

Out[456]= $\left(-\text{peq} + 2\,\text{p}\right)\left(2 - \frac{2\,\text{M}}{\text{r}}\right) + \sqrt{3}\left(\frac{1}{1 - \frac{2\,\text{M}}{\text{r}}} - \frac{2\,\text{M}}{3\,\text{r}}\right)\text{S} + \rho$

We can make a command to do this,

In[457]:=
```
SETV[α_, β_] := Tvf[{-α, cb}, {-β, cb}]
```

In[459]:=
```
SETV[0, 0]
```

Out[459]= $\left(-\text{peq} + 2\,\text{p}\right)\left(2 - \frac{2\,\text{M}}{\text{r}}\right) + \sqrt{3}\left(\frac{1}{1 - \frac{2\,\text{M}}{\text{r}}} - \frac{2\,\text{M}}{3\,\text{r}}\right)\text{S} + \rho$

The Stress-Energy Tensor for a Viscous Fluid IV

In[460]:=
```
SETVTable = Table[Row[{"T"Grid[{{α,β}}], "→", SETV[α, β]}], {α, 0, 3}, {β, 0, 3}] // TableForm
```

Out[460]//TableForm=

$T_{0\ 0} \to (-peq + 2p)\left(2 - \frac{2M}{r}\right) + \sqrt{3}\left(\frac{1}{1-\frac{2M}{r}} - \frac{2M}{3r}\right) S + \rho$	$T_{0\ 1} \to q1$	$T_{0\ 2} \to q2$
$T_{1\ 0} \to q1$	$T_{1\ 1} \to \frac{-peq+2p}{1-\frac{2M}{r}} + \frac{S}{\sqrt{3}\left(1-\frac{2M}{r}\right)}$	$T_{1\ 2} \to 0$
$T_{2\ 0} \to q2$	$T_{2\ 1} \to 0$	$T_{2\ 2} \to -(-peq + 2p)\ r^2$
$T_{3\ 0} \to q3$	$T_{3\ 1} \to 0$	$T_{3\ 2} \to 0$

Using the Source Conservation Law to Derive the Equations of a Viscous Fluid I

We can write the covariant derivative of the stress-energy tensor.

In[468]:=
```
cdstpfv[α_, β_, γ_] := cdg[-α][Tvf[β, γ]]
```

We can write this,

In[469]:=
```
cdstpfv[α, β, γ] // ToCanonical // Simplify
```

Out[469]=

$$\begin{pmatrix} \begin{pmatrix} \bullet \\ \bullet \\ \bullet \\ \bullet \end{pmatrix} & \begin{pmatrix} \bullet \\ \mathcal{D}_r\, q1 \\ -\frac{q2}{r}+\mathcal{D}_\theta\, q1 \\ -\frac{q3}{2\, r}+\mathcal{D}_\phi\, q1 \end{pmatrix} & \begin{pmatrix} \mathcal{D}_t\, q2 \\ \frac{q2\, (-M+r)}{(2\, M-r)\, r}+\mathcal{D}_r\, q2 \\ 2\, M\, q1-q1\, r+\mathcal{D}_\theta\, q2 \\ -q3\, \mathrm{Cot}[\theta]+\mathcal{D}_\phi\, q2 \end{pmatrix} & \begin{pmatrix} \mathcal{D}_t\, q3 \\ \frac{q3}{4\, M-2\, r}+\mathcal{D}_r\, q3 \\ -q3\, \mathrm{Cot}[\theta]+\mathcal{D}_\theta\, q3 \\ \bullet \end{pmatrix} \\ \begin{pmatrix} \bullet \\ \mathcal{D}_r\, q1 \\ -\frac{q2}{r}+\mathcal{D}_\theta\, q1 \\ -\frac{q3}{2\, r}+\mathcal{D}_\phi\, q1 \end{pmatrix} & \begin{pmatrix} \bullet \\ \bullet \\ \bullet \\ \bullet \end{pmatrix} & \begin{pmatrix} \frac{M\, q2}{2\, M\, r-r^2} \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \frac{M\, q3}{2\, M\, r-r^2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} \mathcal{D}_t\, q2 \\ \frac{q2\, (-M+r)}{(2\, M-r)\, r}+\mathcal{D}_r\, q2 \\ 2\, M\, q1-q1\, r+\mathcal{D}_\theta\, q2 \\ -q3\, \mathrm{Cot}[\theta]+\mathcal{D}_\phi\, q2 \end{pmatrix} & \begin{pmatrix} \frac{M\, q2}{2\, M\, r-r^2} \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \bullet \\ \bullet \\ \bullet \\ \bullet \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} \mathcal{D}_t\, q3 \\ \frac{q3}{4\, M-2\, r}+\mathcal{D}_r\, q3 \\ -q3\, \mathrm{Cot}[\theta]+\mathcal{D}_\theta\, q3 \\ \bullet \end{pmatrix} & \begin{pmatrix} \frac{M\, q3}{2\, M\, r-r^2} \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \bullet \\ \bullet \\ \bullet \\ \bullet \end{pmatrix} \end{pmatrix} \beta\gamma\alpha$$

We can specify the specific components of this. Here we have the covariant derivative of the time-time component.

In[470]:=
```
cdstpfv[{0, cb}, {0, cb}, {0, cb}] // ToCanonical // Simplify
```

Out[470]=

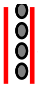
$$\left(4-\frac{4\, M}{r}\right)\, (\mathcal{D}_t\, p)+\frac{\left(-\frac{2\, M}{r}-\frac{3\, r}{2\, M-r}\right)\, (\mathcal{D}_t\, S)}{\sqrt{3}}+\mathcal{D}_t\, \rho$$

We can write a program for this,

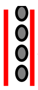
```
In[471]:= cdsetpfv[α_, β_, γ_] := cdstpfv[{α, cb}, {β, cb}, {γ, cb}]

In[473]:= cdsetpfv[0, 0, 0] // ToCanonical // Simplify
```

Out[473]= $$\left(4-\frac{4\,M}{r}\right)(\mathcal{D}_t\,p)+\frac{\left(-\frac{2\,M}{r}-\frac{3\,r}{2\,M-r}\right)(\mathcal{D}_t\,S)}{\sqrt{3}}+\mathcal{D}_t\,\rho$$

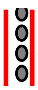
```
In[474]:= cdsetpfv[1, 0, 0] // ToCanonical // Simplify
```

Out[474]=  -1

```
In[475]:= cdsetpfv[2, 0, 0] // ToCanonical // Simplify
```

Out[475]=  -2

```
In[476]:= cdsetpfv[3, 0, 0] // ToCanonical // Simplify
```

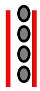
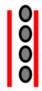
Out[476]=  -3

Using the Source Conservation Law to Derive the Equations of a Viscous Fluid II

Putting together all of the derivatives of the time components

```
In[478]:= tcdsetpftv = Total[Table[cdsetpfv[α, 0, 0], {α, 0, 3}]] // ToCanonical // Simplify
```

Out[478]=

$_{-3}$ + $_{-2}$ + $_{-1}$ $+ 4\,(\mathcal{D}_t\, p) - \frac{4\, M\, (\mathcal{D}_t\, p)}{r} + \frac{\sqrt{3}\,(\mathcal{D}_t\, S)}{1 - \frac{2M}{r}} - \frac{2\, M\, (\mathcal{D}_t\, S)}{\sqrt{3}\, r} + \mathcal{D}_t\, \rho$

Setting this to 0 will give us the continuity equation for a viscous fluid in Schwarzschild spacetime.

Using the Source Conservation Law to Derive the Equations of a Viscous Fluid III

In order to get the equations of momentum we need to apply a projection tensor.

In[484]:=
```
momv[α_, β_, γ_, μ_, ν_] := P[μ, ν] × cdstpfv[α, β, γ] // ToCanonical // Simplify
```

In[485]:=
```
mom[α, β, γ, μ, ν]
```

Out[485]=

```
Row 1, block 1:
{ {●,0,0,0}, {0,●,0,0}, {0,0,-r (-2 M (𝒟t p)+r (𝒟t ρ)),0}, {0,0,0,●} }
{ {●,0,0,0}, {0,●,0,0}, {0,0,●,0}, {0,0,0,●} }
{ {●,0,0,0}, {0,●,0,0}, {0,0,-r (-2 M (𝒟θ p)+r (𝒟θ ρ)),0}, {0,0,0,●} }
{ {●,0,0,0}, {0,●,0,0}, {0,0,-r (-2 M (𝒟ϕ p)+r (𝒟ϕ ρ)),0}, {0,0,0,●} }

Row 1, block 2:
{ {(2 M (M-r) (p-ρ))/((2 M-r) r^2),0,0,0}, {0,(M (p-ρ))/(-2 M+r)^2,0,0}, {0,0,(M r (p-ρ))/(2 M-r),0}, {0,0,0,(2 M Sin[θ]^2 (-p+ρ))/(2 M-r)} }
{ {0,0,0,0}, {0,0,0,0}, {0,0,0,0}, {0,0,0,0} }
{ {0,0,0,0}, {0,0,0,0}, {0,0,0,0}, {0,0,0,0} }
{ {0,0,0,0}, {0,0,0,0}, {0,0,0,0}, {0,0,0,0} }

Row 2, block 1:
{ {(2 M (M-r) (p-ρ))/((2 M-r) r^2),0,0,0}, {0,(M (p-ρ))/(-2 M+r)^2,0,0}, {0,0,(M r (p-ρ))/(2 M-r),0}, {0,0,0,(2 M Sin[θ]^2 (-p+ρ))/(2 M-r)} }
{ {0,0,0,0}, {0,0,0,0}, {0,0,0,0}, {0,0,0,0} }
{ {0,0,0,0}, {0,0,0,0}, {0,0,0,0}, {0,0,0,0} }
{ {0,0,0,0}, {0,0,0,0}, {0,0,0,0}, {0,0,0,0} }

Row 2, block 2:
{ {(2 (M-r) (𝒟t p))/(2 M-r),0,0,0}, {0,(r^2 (𝒟t p))/(-2 M+r)^2,0,0}, {0,0,(r^3 (𝒟t p))/(2 M-r),0}, {0,0,0,(2 r Sin[θ]^2 (𝒟t p))/(1-(2 M)/r)} }
{ {(2 (M-r) (𝒟r p))/(2 M-r),0,0,0}, {0,(r^2 (𝒟r p))/(-2 M+r)^2,0,0}, {0,0,(r^3 (𝒟r p))/(2 M-r),0}, {0,0,0,-((2 r^2 Sin[θ]^2 (𝒟r p))/(2 M-r))} }
{ {(2 (M-r) (𝒟θ p))/(2 M-r),0,0,0}, {0,(r^2 (𝒟θ p))/(-2 M+r)^2,0,0}, {0,0,(r^3 (𝒟θ p))/(2 M-r),0}, {0,0,0,(2 r Sin[θ]^2 (𝒟θ p))/(1-(2 M)/r)} }
{ {(2 (M-r) (𝒟ϕ p))/(2 M-r),0,0,0}, {0,(r^2 (𝒟ϕ p))/(-2 M+r)^2,0,0}, {0,0,(r^3 (𝒟ϕ p))/(2 M-r),0}, {0,0,0,(2 r Sin[θ]^2 (𝒟ϕ p))/(1-(2 M)/r)} }

Row 3, block 1: 4×4 array of {0,0,0,0}
Row 3, block 2: 4×4 array of {0,0,0,0}
Row 3, block 3:
{ {2 (M-r) r (𝒟t p),0,0,0}, {2 (M-r) r (𝒟r p),0,0,0}, {2 (M-r) r (𝒟θ p),0,0,0}, {2 (M-r) r (𝒟ϕ p),0,0,0} }

Row 4, block 1: 4×4 array of {0,0,0,0}
Row 4, block 2: 4×4 array of {0,0,0,0}
```

Once again, you can read the components off the lists to get the specific components.

More work on this needs to be done to extract features like the vorticity tensor.

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Thank You!