On the Cowling Approximation:

A Verification of the Method via Functional and Asymptotic Analysis

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Outline

The system of Equations

Perturbation and Linearization.

We introduce the basic principles [AC-DK, Cor, LW, SVH, T]
Perturbation quantities ξ, η:

- Lagrangian displacement vector $\vec{\xi}$
- $\vec{\xi} = \xi \hat{r} + \nabla_h \eta$ (spherical coords.) ξ is aka δr.
- $\nabla_h \eta = \left(0, \frac{1}{r}\frac{\partial \eta}{\partial \theta}, \frac{1}{r \sin\theta}\frac{\partial \eta}{\partial \phi}\right)$
- Spheroidal normal modes $(\nabla \times \vec{\xi})_r = \vec{0}$

Governing Equations - Isentropic Model.

Physical quantities: ρ density, P pressure, V potential, $\vec{g}$ acceleration due to gravity, Γ_1 is adiabatic exponent.
Linearized Governing Equations: Expressed in terms of $\vec{\xi}$:

- $\delta\rho = -\nabla \cdot (\rho\vec{\xi})$ (mass conservation)
- $\delta P = -\Gamma_1 P \nabla \cdot \vec{\xi} - \vec{\xi} \cdot \nabla P$ (isentropic equation of state)
- $\frac{1}{\rho}\nabla P = \vec{g} = -\nabla V$ (equilibrium)
- $\nabla^2 \delta V = -4\pi G \nabla \cdot (\rho\vec{\xi})$ (Poisson's equation).

(δ Eulerian perturbation; ' derivative wrt r)

- Expansions in terms of spherical harmonics Y_ℓ^m. Superposition of modes of the form

$$e^{i(\sigma_\ell t + m\phi)} \eta_{l,m}(r) Y_\ell^m(\theta, \phi)$$
$$e^{i(\sigma_\ell t + m\phi)} \xi_{l,m}(r) Y_\ell^m(\theta, \phi)$$

- After separation of variables, equations decouple w.r.t ℓ .
- Spheroidal normal modes are degenerate w.r.t. m.

Resulting System - LW Formulation

Subscripts are dropped, understanding that *frequency* σ and dependent variables depend on *degree* ℓ

$$\frac{d\,u}{d\,r} = \frac{g}{c^2}u + \left[\frac{\ell(\ell+1)}{\sigma^2} - \frac{r^2}{c^2}\right]y + \frac{r^2}{c^2}\Phi \tag{1}$$

$$\frac{d\,y}{d\,r} = \frac{\sigma^2 - N^2}{r^2}u + \frac{N^2}{g}y - \frac{d}{d\,r}\Phi(r) \tag{2}$$

$$\frac{1}{r^2}\frac{d}{d\,r}\left(r^2\frac{d\,\Phi}{d\,r}\right) - \frac{\ell(\ell+1)}{r^2}\Phi = 4\pi G\rho\left(\frac{N^2}{r^2 g}u + \frac{1}{c^2}y\right) \tag{3}$$

(Ledoux and Walraven)

Here

- $u \stackrel{\text{def}}{=} r^2\xi$, $y \stackrel{\text{def}}{=} \frac{\delta p}{\rho}$, $\Phi \stackrel{\text{def}}{=} \delta V$
- $c = \sqrt{\frac{\Gamma_1 p}{\rho}}$ is the speed of sound
- $N^2 = -g\left(\frac{g}{c^2} + \frac{d\ln\rho}{d\,r}\right)$: N is the *Brunt-Väisäla frequency*
- Boundary conditions at stellar center $r = 0$ and stellar surface $r = R$, typically.
- Boundary-value problems in terms of σ form (non-linear) eigenvalue problems for each ℓ: σ called an *eigenfrequency.*

Alternative Formulation - ES

$$\frac{d\,u}{d\,r} = \frac{g}{c^2}u + \left[\ell(\ell+1) - \sigma^2\frac{r^2}{c^2}\right]\eta + \frac{r^2}{c^2}\Phi \tag{4}$$

$$\frac{d\,\eta}{d\,r} = \frac{1}{r^2}\left(1 - \frac{N^2}{\sigma^2}\right)u + \frac{N^2}{g}\eta - \frac{N^2}{\sigma^2 g}\Phi(r) \tag{5}$$

$$\frac{1}{r^2}\frac{d}{d\,r}\left(r^2\frac{d\,\Phi}{d\,r}\right) - \left[\frac{\ell(\ell+1)}{r^2} - \frac{4\pi G\rho}{c^2}\right]\Phi = 4\pi G\rho\left(\frac{N^2}{r^2 g}u + \frac{\sigma^2}{c^2}\eta\right) \tag{6}$$

(Eisenfeld and Smeyers)

Cowling Approximation

In the so-called Cowling approximation, the perturbation $\Phi = \delta V$ is neglected in (1), (2) or in (4), (5) leaving

$$\frac{d\,u}{d\,r} = \frac{g}{c^2}u + \left[\frac{\ell(\ell+1)}{\sigma^2} - \frac{r^2}{c^2}\right]y$$

$$\frac{d\,y}{d\,r} = \frac{\sigma^2 - N^2}{r^2}u + \frac{N^2}{g}y$$

or

$$\frac{d\,u}{d\,r} = \frac{g}{c^2}u + \left[\ell(\ell+1) - \sigma^2\frac{r^2}{c^2}\right]\eta$$

$$\frac{d\,\eta}{d\,r} = \frac{1}{r^2}\left(1 - \frac{N^2}{\sigma^2}\right)u + \frac{N^2}{g}\eta$$

(resp.).

Main Technique

We apply analysis of equations of more abstract form

$$\mathcal{L}\vec{x} = F(\vec{x}) + \vec{f} \tag{7}$$

where

- F is symmetric on a real Hilbert space $\mathbb{H}$ (eg. $L^2(a, b)$).
- $\mathcal{L}$ self-adjoint (SA) on a subspace $\mathcal{H}$ dense in $\mathbb{H}$: $\mathcal{H}$ is called a *core* for $\mathcal{L}$.
- $\mathcal{H}$ may depend on SA boundary conditions imposed.
- L^2: Think QM; integral inner product; $||\vec{x}|| = \sqrt{< \vec{x}, \vec{x} >}$

A Theorem by Amann

We will be applying the following theorem which is a special case (linear version) of Theorem 2.6 of [A]

Theorem 2.1

Suppose the following hold for some $\gamma > 0$:

- $\langle(\mathcal{L} - F)\vec{x}, \vec{x}\rangle \leq -\gamma||\vec{x}||^2 \ \forall \vec{x} \in \mathcal{H}_1$
- $\langle(\mathcal{L} - F)\vec{x}, \vec{x}\rangle \geq \gamma||\vec{x}||^2 \ \forall \vec{x} \in \mathcal{H}_2$;
- $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$

Then, there is a unique solution $\vec{y} \in \mathcal{H}$ *to (7); and, the solution satisfies*

$$||\vec{y}|| \leq \frac{2}{\gamma}||\vec{f}||$$

Particulars

$\mathcal{L}$ and F have complete set of eigenfunctions

- $\{\psi_k\}_{k=1}^n$ and $\{\phi_k\}_{k=1}^n$ with associated eigenvalues $\{\lambda_k\}_{k=1}^n$ and $\{\nu_k\}_{k=1}^n$, resp.
- The spectra are discrete (pure-point) where

$$\lambda_k \searrow -\infty; \;\; \nu_k \nearrow 0$$

- Slight modification allows assumption of only non-zero eigenvalues.
- We can take $\gamma = dist(spec(\mathcal{L}), spec(F)) > 0$

Enter Asymptotics

- We see homogeneous systems

$$\frac{d}{dr}X = \sigma^2 AX$$

for large $\sigma > 0$ where $A = \Lambda_0 + \sigma^{-2}\Lambda_1 + \sigma^{-4}\Lambda_2$

- Apply asymptotic methods [CL] in combination with main theorem.
- Form non-homogeneous equations to effectively decouple.
- Change of variables and alternative parameterizations pursued.

Adiabatic Equilibrium: Simplifying Assumption

We will set

$$N^2 = 0$$

on $[a, b]$ *adiabatic equilibrium* whereby

- aka isentropic equilibrium (explicitly assume reversibility)
- $g = -c^2 \frac{d \ln \rho}{d\, r}$
- A very convenient integrating factor results.

Recast ES

The first two equations of the ES system

$$\frac{d\,u}{d\,r} - \frac{g}{c^2}u - \left[\ell(\ell+1) - \sigma^2\frac{r^2}{c^2}\right]\eta = \frac{r^2}{c^2}\Phi \tag{8}$$

$$\frac{d\,\eta}{d\,r} - \frac{1}{r^2}u = 0 \tag{9}$$

Rewrite ES as

$$\frac{d}{d\,r}\left(r^2\rho(r)\frac{d\,\eta}{d\,r}\right) - \left[\ell(\ell+1) - \sigma^2\frac{r^2}{c^2}\right]\rho(r)\eta = \frac{r^2}{c^2}\Phi$$

$$\frac{d}{d\,r}\left(r^2\frac{d\,\Phi}{d\,r}\right) - \left[\ell(\ell+1) - \frac{4\pi G\rho r^2}{c^2}\right]\Phi = 4\pi G\rho\left(\frac{\sigma^2 r^2}{c^2}\eta\right)$$

Imposing SA boundary conditions, the LHS of each equation is of S-L form which we will write as

$$\mathcal{J}_{\ell\sigma}\eta = \frac{r^2}{c^2}\rho\Phi$$

$$\mathcal{L}_\ell\Phi = 4\pi G\rho\left(\frac{\sigma^2 r^2}{c^2}\eta\right)$$

- Operators $\mathcal{J}_{\ell\sigma}$ and $\mathcal{L}_\ell$ are S-L type.
- Combining equations to obtain: $\mathcal{L}_\ell\Phi = \sigma^2 F(\Phi) + \sigma^2 f_0$
- Indeed, F is symmetric as above.

Recast LW

Now set $N = 0$ in the LW system, (1)(2)(3)

$$\frac{d\,u}{d\,r} = \frac{g}{c^2}u + \left[\frac{\ell(\ell+1)}{\sigma^2} - \frac{r^2}{c^2}\right]y + \frac{\ell(\ell+1)}{\sigma^2}\Phi \tag{10}$$

$$\frac{d\,y}{d\,r} = \frac{\sigma^2}{r^2}u - \frac{d\,\Phi}{d\,r} \tag{11}$$

$$\frac{d}{d\,r}\left(r^2\frac{d\,\Phi}{d\,r}\right) - \ell(\ell+1)\Phi = 4\pi G\frac{\rho r^2}{c^2}y \tag{12}$$

Change Variables and Parameters

And, setting $v \stackrel{\text{def}}{=} \frac{1}{r}u$, $\zeta^2 \stackrel{\text{def}}{=} \ell(\ell+1)$ and $\sigma^2 \stackrel{\text{def}}{=} z\zeta$, equations (10) and (11) become

$$\frac{d\,v}{d\,r} = \left(\frac{g}{c^2} - \frac{1}{r}\right) v + \left[\frac{\zeta}{zr} - \frac{r}{c^2}\right] y + \frac{\zeta}{zr}\Phi$$

$$\frac{d\,y}{d\,r} = \frac{\zeta z}{r} v - \frac{d\,\Phi}{d\,r}$$

respectively.

LW in Partial Matrix Form

We then form a system of equations in matrix form $\frac{d}{dr}Y = \zeta AY + \mathfrak{G}$ where

$$Y = \begin{bmatrix} v \\ y \end{bmatrix}, \quad \mathfrak{G} = \begin{bmatrix} \frac{\zeta}{zr}\Phi \\ -\frac{d\Phi}{dr} \end{bmatrix}$$

Then, $A = A_0 + \frac{1}{\zeta}A_1$ for

$$A_0 = \frac{1}{r}\begin{bmatrix} 0 & \frac{1}{z} \\ z & 0 \end{bmatrix} \text{ and } A_1 = \begin{bmatrix} \frac{rg-c^2}{rc^2} & -\frac{r}{c^2} \\ 0 & 0 \end{bmatrix}$$

Much of the analysis methods for ES will also be applied here.

ES Results: Large ℓ, Bounded σ

- For sufficiently large ℓ, there results

$$||\Phi_p|| \leq \frac{2\sigma^2}{\gamma}||f_0|| \leq \sigma^2 \frac{Const.}{\gamma}||\eta_0||$$

- Indeed, $||\Phi_p|| = O(\ell^{-2})$ as $\ell \to \infty$
- General idea: $\Phi \approx \Phi_0$ in norm for large ℓ
- In turn, $\Phi \approx 0$ verifies Cowling.

Large σ; Matrix Form

We apply asymptotic estimates wrt large σ

- Change of variables $w \stackrel{\text{def}}{=} \rho\sigma^{-1}u$ to recast (8) & (9)
- Leads to non-homogeneous system of the form $\vec{Y}' = \sigma A_0 \vec{Y} + \sigma^{-1} A_2 + \sigma^{-1}\vec{f}$
- Apply asymptotic methods together with main estimate
- Check calculations via Mathematica using method related to [W]. (See Appendix.)

Insert Asymptotics

We arrive at a particular solution satisfying

$$\eta_p = \frac{\sqrt{c(r)}}{\sigma r \sqrt{\rho(r)}} \sin(\sigma\theta(r)) \int_a^r t \sqrt{\frac{\rho(t)}{c^3(t)}} \cos(\sigma\theta(t)) \Phi(t)\, dt +$$

$$\frac{\sqrt{c(r)}}{\sigma r \sqrt{\rho(r)}} \cos(\sigma\theta(r)) \int_r^b t \sqrt{\frac{\rho(t)}{c^3(t)}} \sin(\sigma\theta(t)) \Phi(t)\, dt + O(\sigma^{-3})$$

$$\stackrel{\text{def}}{=} \sigma^{-1} \mathcal{W}(\Phi) + O(\sigma^{-3})$$

for $\theta(r) \stackrel{\text{def}}{=} \int^r \frac{1}{c(r)}\, dr$

Sharper Estimate

We can sharpen the estimate if a symmetric F can be found:

- $\frac{r^2\rho}{c^2}\mathcal{W}$ is symmetric
- $\gamma \geq \alpha$ and $\alpha = O(\sigma^2)$ as $\sigma \to +\infty$
- $\frac{\sigma^2}{\gamma} = O(1)$
- Obtain
$$||\Phi_p|| \leq C_1 \frac{1}{\sigma\gamma} + C_2||\eta_0||$$
with $\gamma^{-1} = O(\sigma^{-2})$
- Since θ' is positive on $[a, b]$, we find $||\eta_0|| = O(\sigma^{-1})$

LW: ℓ and σ^2 Large but Comparable.

We find a particular y with estimates uniform r and ζ: $y_p =$

$$\frac{1}{2}[r^{\zeta-1/2}e^{\mathcal{I}^-(r)}\int_a^r \Phi(t)(t^{-\zeta+1/2}e^{-\mathcal{I}^-(t)})'dt +$$

$$r^{-\zeta-1/2}e^{\mathcal{I}^+(r)}\int_r^b \Phi(t)(t^{\zeta+1/2}e^{-\mathcal{I}^+(t)})'dt] - \Phi(r) + O(\zeta^{-2})$$

(as $\zeta \to +\infty$) for $\mathcal{I}^{\pm}(r) \stackrel{\text{def}}{=} \int \frac{g \pm zr}{c^2}\, dr$, respectively.

Finding a Symmetric Operator

We notice that if $z\frac{r}{c^2}$, $\frac{\rho'}{\rho}$ are small (say z small and $\ln\rho$ slowly varying) compared to $\frac{1}{r}$, then y_p can be written as

$$y_p = \int_a^b W(r,t)\Phi(t)dt - \Phi(r) + \epsilon O(\zeta^{-1}) + O(\zeta^{-2})$$

where $W(r,t)$ is a symmetric kernel.
Here, ϵ is small if $\mathcal{I}^{\pm}$ and their derivatives are uniformly small.

Case for Sharper Estimates

Then,

- Consider

$$\frac{\rho r^2}{c^2} = \frac{\rho(r_0) r_0^2}{c^2(r_0)} + O(|r - r_0|)$$

- equation (6) becomes of the familiar general form

$$\mathcal{L}\Phi = F(\Phi) + f_0$$

- resulting in $||\Phi_p|| \leq \frac{1}{\gamma}\left(C_1 + C_2(\zeta^{-1}) + C_3||\eta_0||\right)$
- with $\gamma^{-1} = O(\zeta^{-2})$.
- C_1 is small if $\frac{\rho r^2}{c^2}$ is slowly varying on (a, b).

A Relativistic Pulsation Model

Taking from Lindblom, Mendell, Ipser [LMI]: Regge-Wheeler guage whereby the metric perturbation is given by

$$(g_{ab} + \delta g_{ab}) dx^a dx^b =$$

$$-e^{\nu}(1 - H_0 Y_\ell^m e^{i\sigma t})dt^2 + 2iH_1 Y_\ell^m e^{i\sigma t} dt\, dr + e^{\lambda}(1 - H_0 Y_\ell^m e^{i\sigma t})dr^2$$
$$+r^2(1 - K Y_\ell^m e^{i\sigma t})(d\theta^2 + \sin^2\theta d\phi^2)$$

Barotropic (Eulerian) perturbations

$$\delta\rho = \frac{d\rho}{dP}\delta P\, Y_m^\ell e^{i\sigma t}$$

Speed of light and G equal to 1, even parity

- H_1 and K expressed in terms of H_0 and $\delta U = \frac{\delta P}{\rho + P} + H_0/2$
- Transformation involved generally may have singularities
- Consider weak field and slow rotation.

$$\delta U'' + \left(\frac{2}{r} - \frac{\nu'}{2}\frac{d\rho}{dP} + v_1\right)\delta U' + \left[\frac{\sigma^2}{e^\nu}\frac{d\rho}{dP} - \frac{\ell(\ell+1)}{r^2} + v_2\right] e^\lambda \delta U$$

$$= v_3 H_0' + \left[\frac{\sigma^2}{2e^\nu}\frac{d\rho}{dP} + v_4\right] e^\lambda H_0;$$

$$H_0'' + \left(\frac{2}{r} + \eta_1\right) H_0' + \left[\frac{\sigma^2}{e^\nu} - \frac{\ell(\ell+1)}{r^2} + 4\pi(P+\rho)\frac{d\rho}{dP} + \eta_2\right] e^\lambda H_0$$

$$= \eta_3 \delta U' + \left[8\pi(P+\rho)\frac{d\rho}{dP} + \eta_4\right] e^\lambda \delta U$$

- $\frac{M}{R}, \sigma, \frac{P}{\rho} < \epsilon << 1$
- Replace $H_0 = 2\Phi$ and $\delta U = 2\sigma^2\eta$
- Modulo $O(\epsilon)$: $\nu' = -2\frac{P'}{\rho}$, $P' = -\rho g$, $\frac{d\rho}{dP} = 1/c^2$
- $v_j = \eta_j = O(\epsilon)$

$$\mathcal{L}_\ell\Phi + (v_1 + O(\epsilon))\Phi' + (v_2 + O(\epsilon))\Phi = v_3\eta' + (\frac{4\pi\sigma^2 r^2\rho}{c^2} + v_4 + O(\epsilon))\eta$$

$$\mathcal{J}_\ell\eta + (\eta_1 + O(\epsilon))\eta' + (\eta_2 + O(\epsilon))\eta = \eta_3\Phi' + (\frac{r^2\rho}{c^2} + \eta_4 + O(\epsilon))\Phi$$

ES Formulation to Order ϵ

Integrating factors $e^{\nu_1}\frac{r^2}{\rho}$ and $e^{\eta_1}\frac{r^2}{\rho}$ respectively yield

$$\mathcal{J}^\sharp \eta = (\frac{r^2\rho}{c^2} + O(\epsilon))\Phi + O(\epsilon)\Phi'$$

$$\mathcal{L}^\sharp \Phi = (4\pi\sigma^2\frac{r^2\rho}{c^2} + O(\epsilon))\eta + O(\epsilon)\eta'$$

to form

$$\mathcal{L}^\sharp \Phi = \sigma^2 F^\sharp(\Phi) + \sigma^2 f_0$$

with $\mathcal{L}^\sharp$ S-L, $F^\sharp$ symmetric, and $||f_0|| \leq C||\eta_0|| + O(\epsilon)||\Phi||$
Likewise,

$$||\Phi|| \leq \frac{\sigma^2 Const}{\gamma}||\eta_0||$$

Summary

- The Cowling approximation appears to be verified under certain conditions in the case of adiabatic equilibrium.
- We find particular Φ_p small in norm for large degree compared to certain other homogeneous dependent variables.
- Analysis can apply to relativistic pulsation if approximately Newtonian.

Outlook

Continuation:

- Search for failures of Cowling approximation: Use method in reverse to find Φ_p relatively large.
- Extend to cases of stable equilibrium $N^2 > 0$ (against convection) - perhaps a perturbation of present case.
- What if eigenvalues are nested?
- Study spectral distance and resulting estimates as they depend on length of interval (a, b).
- Involve various order-of-magnitude estimates of physical quantities: Determine practicalities of method.
- Some non-linear pulsation models may perhaps be investigated by further application of results of [A].

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Below are Mathematica works involving is asymptotic estimates used.

Asymptotics for Large parameter in a Newtonian Stellar Pulsation Model 1:

We develop asymptotic estimates for a system of the form

$Y'[t] = \rho \mathcal{A}_0 Y + \mathcal{A}_1 Y + \rho^{-1} \mathcal{A}_0 Y_2$

for large real parameter ρ. The matrices in this case are given:

In[261]:=
```
𝒜0 = t^(-1) * {{0, 1/z}, {z, 0}};
MatrixForm[%]
```

Out[262]//MatrixForm=

$$\begin{pmatrix} 0 & \frac{1}{t z} \\ \frac{z}{t} & 0 \end{pmatrix}$$

In[263]:=
```
𝒜1 = {{(t * g[t] - (c[t])^2) / (t * (c[t])^2), -t / (c[t])^2}, {0, 0}};
MatrixForm[%]
```

Out[264]//MatrixForm=

$$\begin{pmatrix} \frac{-c[t]^2+t\,g[t]}{t\,c[t]^2} & -\frac{t}{c[t]^2} \\ 0 & 0 \end{pmatrix}$$

In[265]:=
```
𝒜2 = {{0, 0}, {0, 0}};
MatrixForm[%]
```

Out[266]//MatrixForm=

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

We now diagonalize the leading matrix and convert the differential system to the form

$X' = \rho A_0 X + A_1 X + \rho^{-1} A_2 X$ for $Y = PX$.

In[267]:=
```
{P, A0} = JordanDecomposition[𝒜0]
A1 = Inverse[P].𝒜1.P - Inverse[P].D[P, t]
A2 = Inverse[P].𝒜2.P;
```

Out[267]= $\left\{\left\{\left\{-\frac{1}{z}, \frac{1}{z}\right\}, \{1, 1\}\right\}, \left\{\left\{-\frac{1}{t}, 0\right\}, \left\{0, \frac{1}{t}\right\}\right\}\right\}$

Out[268]= $\left\{\left\{\frac{t\,z}{2\,c[t]^2} + \frac{-c[t]^2+t\,g[t]}{2\,t\,c[t]^2}, \frac{t\,z}{2\,c[t]^2} - \frac{-c[t]^2+t\,g[t]}{2\,t\,c[t]^2}\right\}, \left\{-\frac{t\,z}{2\,c[t]^2} - \frac{-c[t]^2+t\,g[t]}{2\,t\,c[t]^2}, -\frac{t\,z}{2\,c[t]^2} + \frac{-c[t]^2+t\,g[t]}{2\,t\,c[t]^2}\right\}\right\}$

Our goal is to follow [CL] to develop asymptotic estimates for a fundamental solution M for the in the form

$\mathcal{F} = e^{\rho Q_0 + Q_1}\left(I + \rho^{-1} P_1 + \rho^{-2} P_1\right)$

The matrices in the exponent are diagonal where none of the Q's or P's depend on parameter ρ. The procedure, broadly speaking, is to solve
the differential equation formally in equating terms of formal series in the parameter ρ in substituting into the asymptotic expression into the
differential equation: Off-diagonal terms (Offdiag[]) and on-diagonal (Ondiag[]) terms for the P's are solve separately. To do this we produce matrices with undetermined coefficients and solve for them either via Solve[] or DSolve[] and substitute the solutions accordingly. We will impose a condition at $t = 1$ in our integration steps.

In[270]:=
```
Pmtc[n_] := Array[Pels, {4, 2, 2}][[n]]
Pmtc0 = IdentityMatrix[2]
Qpmtx[n_] := DiagonalMatrix[Array[Qprm, {2, 1, 2}][[n]][[1]]]
Offdiag[a_] := a - DiagonalMatrix[Diagonal[a]]
Ondiag[a_] := DiagonalMatrix[Diagonal[a]]
```

Out[271]= {{1, 0}, {0, 1}}

We solve the off-diagonal terms of P_1 and the Q_0 terms:

In[275]:=
```
Q0 = Integrate[A0, t]
LHS0 = Pmtc0.Qpmtx[1] + Pmtc[1].A0;
RHS0 = A0.Pmtc[1] + A1.Pmtc0;
sol1 = Solve[{Offdiag[LHS0] == Offdiag[RHS0]},
   Complement[Flatten[Pmtc[1]], Diagonal[Pmtc[1]]]];
NewP1els[i_, j_] := If[i ≠ j, Pmtc[1][[i, j]] //. sol1[[1]], Pmtc[1][[i, j]][t]];
NewPmtx = Array[NewP1els, {2, 2}]
SolQ1 = Solve[Ondiag[LHS0] == Ondiag[RHS0], Diagonal[Qpmtx[1]]]
NewQmtx1 = Qpmtx[1] //. SolQ1[[1]];
Q1 = Integrate[NewQmtx1, t]
```

Out[275]= {{-Log[t], 0}, {0, Log[t]}}

Out[280]= $\left\{\left\{\text{Pels}[1,1,1][t],\ \frac{t^2 z + c[t]^2 - t\,g[t]}{4\,c[t]^2}\right\},\ \left\{\frac{t^2 z - c[t]^2 + t\,g[t]}{4\,c[t]^2},\ \text{Pels}[1,2,2][t]\right\}\right\}$

Out[281]= $\left\{\left\{\text{Qprm}[1,1,1] \to \frac{t^2 z - c[t]^2 + t\,g[t]}{2\,t\,c[t]^2},\ \text{Qprm}[1,1,2] \to \frac{-t^2 z - c[t]^2 + t\,g[t]}{2\,t\,c[t]^2}\right\}\right\}$

Out[283]= $\left\{\left\{\frac{1}{2}\int \frac{t^2 z - c[t]^2 + t\,g[t]}{t\,c[t]^2}\,dt,\ 0\right\},\ \left\{0,\ \frac{1}{2}\int \frac{-t^2 z - c[t]^2 + t\,g[t]}{t\,c[t]^2}\,dt\right\}\right\}$

We now complete P_1 and solve the off-diagonal terms for P_2:

In[284]:=
```
LHS1 = D[NewPmtx, t] + NewPmtx.NewQmtx1 + Pmtc[2].A0;
RHS1 = A1.NewPmtx + A0.Pmtc[2] + A2;
eqn1 = Simplify[Diagonal[LHS1] - Diagonal[RHS1]];
initvals = Diagonal[NewPmtx] //. t → 1;
sol1b = DSolve[{eqn1 == {0, 0}, initvals == {0, 0}}, Diagonal[NewPmtx], t];
P1 = NewPmtx //. Flatten[sol1b]
Sol2 = Solve[Offdiag[LHS1] == Offdiag[RHS1],
     Complement[Flatten[Pmtc[2]], Diagonal[Pmtc[2]]]] //. sol1b[[1]] //. sol1[[1]];
NewP1els2[i_, j_] := If[i ≠ j, Pmtc[2][[i, j]] //. Flatten[Sol2], Pmtc[2][[i, j]][t]];
NewPmtx2 = Array[NewP1els2, {2, 2}];
```

Out[289]=

$$\left\{\left\{\int_1^t \frac{-c[K[1]]^4 + 2\,c[K[1]]^2\, g[K[1]] \times K[1] - g[K[1]]^2\, K[1]^2 + z^2\, K[1]^4}{8\, c[K[1]]^4\, K[1]} \, dK[1],\ \frac{t^2 z + c[t]^2 - t\, g[t]}{4\, c[t]^2}\right\},\ \left\{\frac{t^2 z - c[t]^2 + t\, g[t]}{4\, c[t]^2},\ \int_1^t \frac{c[K[2]]^4 - 2\, c[K[2]]^2\, g[K[2]] \times K[2] + g[K[2]]^2\, K[2]^2 - z^2\, K[2]^4}{8\, c[K[2]]^4\, K[2]} \, dK[2]\right\}\right\}$$

We complete P_2 as we find the diagonal terms. We have introduced a matrix P_3 but we do not compute any of the since any terms involve them on the diagonals cancel from the equation.

In[293]:=
```
LHS2 = D[NewPmtx2, t] + NewPmtx2.D[Q1, t] + Pmtc[3].A0;
RHS2 = A1.NewPmtx2 + A0.Pmtc[3] + A2.P1;
eqn2 = Simplify[Diagonal[LHS2] - Diagonal[RHS2]];
initvals2 = Diagonal[NewPmtx2] //. t → 1;
sol2b = DSolve[{eqn2 == {0, 0}, initvals2 == {0, 0}}, Diagonal[NewPmtx2], t];
P2 = NewPmtx2 //. Flatten[sol2b];
```

As a test of our work thus far, we substitute the expression into the derived equation. The difference of the two sides should be of order $O(\rho^{-2})$

In[299]:=
```
Formal = (Pmtc0 + ρ^(-1) P1 + ρ^(-2) P2).MatrixExp[ρ * Q0 + Q1];
Series[Simplify[D[Formal, t] - (ρ * A0 + A1 + ρ^(-1) A2).Formal], {ρ, Infinity, 1}]
Series[Simplify[D[Formal, t] - (ρ * A0 + A1 + ρ^(-1) A2).Formal], ρ → 0];
```

Out[300]=

$$\left\{\left\{0,\ e^{\frac{1}{2}\text{Integrate}\left[-\frac{1}{t}+\frac{-t z+g[t]}{c[t]^2},t,\text{Assumptions}\to\text{Re}[\rho]>4096\&\&-\frac{1}{4096}<\text{Im}[\rho]<\frac{1}{4096}\right]+\left(\text{Log}[t]\,\rho-\text{Log}[t]+O\left[\frac{1}{\rho}\right]^2\right)}\ O\left[\frac{1}{\rho}\right]^2\right\},\ \left\{e^{\frac{1}{2}\left(\text{Integrate}\left[-\frac{1}{t}+\frac{t z+g[t]}{c[t]^2},t,\text{Assumptions}\to\text{Re}[\rho]>4096\&\&-\frac{1}{4096}<\text{Im}[\rho]<\frac{1}{4096}\right]+\left(-2\,\text{Log}[t]\,\rho-2\,\text{Log}[t]+O\left[\frac{1}{\rho}\right]^2\right)\right)}\ O\left[\frac{1}{\rho}\right]^2,\ 0\right\}\right\}$$

We also test the formal solution by verifying that coefficients canc ρ^{-k} in the differential equation for $k = 1, 2$ which involve the solved terms.

In[302]:=
```
TestLHS1 = D[P1, t] + P1.D[Q1, t] + P2.A0;
TestRHS1 = A1.P1 + A0.P2 + A2;
Testeqn1 = Simplify[TestLHS1 - TestRHS1]
```

Out[304]= {{0, 0}, {0, 0}}

```
In[305]:= P3 = Pmtc[3];
TestLHS2 = Ondiag[D[P2, t] + P2.D[Q1, t] + P3.D[Q0, t]];
TestRHS2 = Ondiag[A1.P2 + A0.P3 + A2.P1];
Simplify[TestLHS2 - TestRHS2]

Out[308]= {{0, 0}, {0, 0}}
```

We obtain our asymptotic estimate P$\mathcal{F}$ for the original system and list the corresponding exponential terms along with correction terms PP_i:

```
In[309]:= Asympt = P.Formal;

In[310]:= Coefficient[Asympt, ρ, 0]
Coefficient[Asympt, ρ, -1] / Coefficient[Asympt, ρ, 0];
Coefficient[Asympt, ρ, -2] / Coefficient[Asympt, ρ, 0];
```

Out[310]= $\left\{\left\{-\frac{e^{\frac{1}{2}\int \frac{t^2 z - c[t]^2 + t g[t]}{t c[t]^2} dt}\, t^{-\rho}}{z}, \frac{e^{\frac{1}{2}\int \frac{-t^2 z - c[t]^2 + t g[t]}{t c[t]^2} dt}\, t^{\rho}}{z}\right\}, \left\{e^{\frac{1}{2}\int \frac{t^2 z - c[t]^2 + t g[t]}{t c[t]^2} dt}\, t^{-\rho}, e^{\frac{1}{2}\int \frac{-t^2 z - c[t]^2 + t g[t]}{t c[t]^2} dt}\, t^{\rho}\right\}\right\}$

Asymptotics for Large parameter In a Newtonian Stellar Pulsation Model 2:

We develop asymptotic estimates for a system of the form

$Y'[t] = \sigma \mathcal{A}_0 Y + \mathcal{A}_1 Y + \sigma^{-1} \mathcal{A}_2 Y$

for large real parameter σ. The matrices in this case are given:

In[209]:=
```
𝒜0 = {{0, -ρ[t] t^2 / c[t]^2}, {1 / (ρ[t] t^2), 0}};
MatrixForm[%]
```

Out[210]//MatrixForm=

$$\begin{pmatrix} 0 & -\frac{t^2 \rho[t]}{c[t]^2} \\ \frac{1}{t^2 \rho[t]} & 0 \end{pmatrix}$$

In[211]:=
```
𝒜1 = {{0, 0}, {0, 0}};
MatrixForm[%]
```

Out[212]//MatrixForm=

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

In[213]:=
```
𝒜2 = {{0, ρ[t] * L}, {0, 0}};
MatrixForm[%]
```

Out[214]//MatrixForm=

$$\begin{pmatrix} 0 & L\rho[t] \\ 0 & 0 \end{pmatrix}$$

We now diagonalize the leading matrix and convert the differential system to the form

$X' = \sigma A_0 X + A_1 X + \sigma^{-1} A_2 X$

for $Y = PX$.

In[215]:=
```
{P, A0} = JordanDecomposition[𝒜0]
A1 = Inverse[P].𝒜1.P - Inverse[P].D[P, t];
A2 = Inverse[P].𝒜2.P;
```

Out[215]= $\left\{\left\{\left\{-\frac{i\, t^2 \rho[t]}{c[t]}, \frac{i\, t^2 \rho[t]}{c[t]}\right\}, \{1, 1\}\right\}, \left\{\left\{-\frac{i}{c[t]}, 0\right\}, \left\{0, \frac{i}{c[t]}\right\}\right\}\right\}$

Our goal is to follow [CL] to develop asymptotic estimates for a fundamental solution M for the in the form

$\mathcal{F} = e^{\sigma Q_0 + Q_1} \left(I + \sigma^{-1} P_1 + \sigma^{-2} P_2\right)$

The matrices in the exponent are diagonal where none of the Q's or P's depend on parameter σ. The procedure is, broadly speaking, is to solve
the differential equation formally in equating terms of formal series in the parameter σ in substituting

into the asymptotic expression into the
differential equation: Off-diagonal terms (Offdiag[]) and on-diagonal (Ondiag[]) terms for the P's are solve separately. To do this we produce matrices with
undetermined coefficients and solve for them either via Solve[] or DSolve[] and substitute the solutions accordingly. We will impose a condition at $t = 1$ in our
integration steps.

In[218]:=
```
Pmtc[n_] := Array[Pels, {4, 2, 2}][[n]]
Pmtc0 = IdentityMatrix[2]
Qpmtx[n_] := DiagonalMatrix[Array[Qprm, {2, 1, 2}][[n]][[1]]]
Offdiag[a_] := a - DiagonalMatrix[Diagonal[a]]
Ondiag[a_] := DiagonalMatrix[Diagonal[a]]
```

Out[219]= {{1, 0}, {0, 1}}

We solve the off-diagonal terms of P_1 and the Q_0 terms:

In[223]:=
```
Q0 = Integrate[A0, t]
LHS0 = Pmtc0.Qpmtx[1] + Pmtc[1].A0;
RHS0 = A0.Pmtc[1] + A1.Pmtc0;
sol1 = Solve[{Offdiag[LHS0] == Offdiag[RHS0]},
   Complement[Flatten[Pmtc[1]], Diagonal[Pmtc[1]]]];
NewP1els[i_, j_] := If[i ≠ j, Pmtc[1][[i, j]] //. sol1[[1]], Pmtc[1][[i, j]][t]];
NewPmtx = Array[NewP1els, {2, 2}]
SolQ1 = Solve[Ondiag[LHS0] == Ondiag[RHS0], Diagonal[Qpmtx[1]]]
NewQmtx1 = Qpmtx[1] //. SolQ1[[1]];
Q1 = Integrate[NewQmtx1, t]
```

Out[223]= $\left\{\left\{-i \int \frac{1}{c[t]}\, dt, 0\right\}, \left\{0, i \int \frac{1}{c[t]}\, dt\right\}\right\}$

Out[228]= $\left\{\left\{\text{Pels}[1,1,1][t], \frac{i\left(-2\, c[t] \times \rho[t] + t\, \rho[t]\, c'[t] - t\, c[t]\, \rho'[t]\right)}{4\, t\, \rho[t]}\right\}, \left\{\frac{i\left(2\, c[t] \times \rho[t] - t\, \rho[t]\, c'[t] + t\, c[t]\, \rho'[t]\right)}{4\, t\, \rho[t]}, \text{Pels}[1,2,2][t]\right\}\right\}$

Out[229]= $\left\{\left\{\text{Qprm}[1,1,1] \to \frac{-2\, c[t] \times \rho[t] + t\, \rho[t]\, c'[t] - t\, c[t]\, \rho'[t]}{2\, t\, c[t] \times \rho[t]}, \text{Qprm}[1,1,2] \to \frac{-2\, c[t] \times \rho[t] + t\, \rho[t]\, c'[t] - t\, c[t]\, \rho'[t]}{2\, t\, c[t] \times \rho[t]}\right\}\right\}$

Out[231]= $\left\{\left\{-\text{Log}[t] + \frac{1}{2}\text{Log}[c[t]] - \frac{1}{2}\text{Log}[\rho[t]], 0\right\}, \left\{0, -\text{Log}[t] + \frac{1}{2}\text{Log}[c[t]] - \frac{1}{2}\text{Log}[\rho[t]]\right\}\right\}$

We now complete $P_1$and solve the off-diagonal terms for P_2:

In[232]:=
```
LHS1 = D[NewPmtx, t] + NewPmtx.NewQmtx1 + Pmtc[2].A0;
RHS1 = A1.NewPmtx + A0.Pmtc[2] + A2;
eqn1 = Simplify[Diagonal[LHS1] - Diagonal[RHS1]];
initvals = Diagonal[NewPmtx] //. t → 1;
sol1b = DSolve[{eqn1 == {0, 0}, initvals == {0, 0}}, Diagonal[NewPmtx], t];
P1 = NewPmtx //. Flatten[sol1b]
Sol2 = Solve[Offdiag[LHS1] == Offdiag[RHS1],
     Complement[Flatten[Pmtc[2]], Diagonal[Pmtc[2]]]] //. sol1b[[1]] //. sol1[[1]];
NewP1els2[i_, j_] := If[i ≠ j, Pmtc[2][[i, j]] //. Flatten[Sol2], Pmtc[2][[i, j]][t]];
NewPmtx2 = Array[NewP1els2, {2, 2}];
```

Out[237]=

$$\Big\{\Big\{\int_1^t \Big(i\,\big(4\,c[K[1]]^2\,\rho[K[1]]^2 + 4\,L\,c[K[1]]^2\,\rho[K[1]]^2 - 4\,c[K[1]]\times K[1]\,\rho[K[1]]^2\,c'[K[1]] + K[1]^2\,\rho[K[1]]^2\,c'[K[1]]^2 + 4\,c[K[1]]^2\,K[1]\times\rho[K[1]]\,\rho'[K[1]] - 2\,c[K[1]]\,K[1]^2\,\rho[K[1]]\,c'[K[1]]\,\rho'[K[1]] + c[K[1]]^2\,K[1]^2\,\rho'[K[1]]^2\big)\Big)\Big/\big(8\,c[K[1]]\,K[1]^2\,\rho[K[1]]^2\big)\,dK[1],\ \frac{i\,\big(-2\,c[t]\times\rho[t] + t\,\rho[t]\,c'[t] - t\,c[t]\,\rho'[t]\big)}{4\,t\,\rho[t]}\Big\},$$

$$\Big\{\frac{i\,\big(2\,c[t]\times\rho[t] - t\,\rho[t]\,c'[t] + t\,c[t]\,\rho'[t]\big)}{4\,t\,\rho[t]},\ \int_1^t \Big(i\,\big(-4\,c[K[2]]^2\,\rho[K[2]]^2 - 4\,L\,c[K[2]]^2\,\rho[K[2]]^2 + 4\,c[K[2]]\times K[2]\,\rho[K[2]]^2\,c'[K[2]] - K[2]^2\,\rho[K[2]]^2\,c'[K[2]]^2 - 4\,c[K[2]]^2\,K[2]\times\rho[K[2]]\,\rho'[K[2]] + 2\,c[K[2]]\,K[2]^2\,\rho[K[2]]\,c'[K[2]]\,\rho'[K[2]] - c[K[2]]^2\,K[2]^2\,\rho'[K[2]]^2\big)\Big)\Big/\big(8\,c[K[2]]\,K[2]^2\,\rho[K[2]]^2\big)\,dK[2]\Big\}\Big\}$$

We complete P_2 as we find the diagonal terms. We have introduced a matrix P_3 but we do not compute any of the since any terms involve them on the diagonals cancel from the equation.

In[241]:=
```
LHS2 = D[NewPmtx2, t] + NewPmtx2.D[Q1, t] + Pmtc[3].A0;
RHS2 = A1.NewPmtx2 + A0.Pmtc[3] + A2.P1;
eqn2 = Simplify[Diagonal[LHS2] - Diagonal[RHS2]];
initvals2 = Diagonal[NewPmtx2] //. t → 1;
sol2b = DSolve[{eqn2 == {0, 0}, initvals2 == {0, 0}}, Diagonal[NewPmtx2], t];
P2 = NewPmtx2 //. Flatten[sol2b];
```

As a test of our work thus far, we substitute the expression into the derived equation. The difference of the two sides should be of order $O(\sigma^{-2})$

In[247]:=
```
Formal = (Pmtc0 + σ^(-1) P1 + σ^(-2) P2).MatrixExp[σ * Q0 + Q1];
Series[Simplify[D[Formal, t] - (σ * A0 + A1 + σ^(-1) A2).Formal], {σ, Infinity, 1}]
Series[Simplify[D[Formal, t] - (σ * A0 + A1 + σ^(-1) A2).Formal], σ → 0];
```

Out[248]=

$$\Big\{\Big\{e^{\text{Integrate}\left[\frac{1}{c[t]},t,\text{Assumptions}\to\text{Re}[\sigma]>4096\&\&-\frac{1}{4096}<\text{Im}[\sigma]<\frac{1}{4096}\right]\left(-i\,\sigma+O\left[\frac{1}{\sigma}\right]^2\right)}\,O\left[\frac{1}{\sigma}\right]^3,$$

$$e^{\text{Integrate}\left[\frac{1}{c[t]},t,\text{Assumptions}\to\text{Re}[\sigma]>4096\&\&-\frac{1}{4096}<\text{Im}[\sigma]<\frac{1}{4096}\right]\left(i\,\sigma+O\left[\frac{1}{\sigma}\right]^2\right)}\,O\left[\frac{1}{\sigma}\right]^2\Big\},$$

$$\Big\{e^{\text{Integrate}\left[\frac{1}{c[t]},t,\text{Assumptions}\to\text{Re}[\sigma]>4096\&\&-\frac{1}{4096}<\text{Im}[\sigma]<\frac{1}{4096}\right]\left(-i\,\sigma+O\left[\frac{1}{\sigma}\right]^2\right)}\,O\left[\frac{1}{\sigma}\right]^2,$$

$$e^{\text{Integrate}\left[\frac{1}{c[t]},t,\text{Assumptions}\to\text{Re}[\sigma]>4096\&\&-\frac{1}{4096}<\text{Im}[\sigma]<\frac{1}{4096}\right]\left(i\,\sigma+O\left[\frac{1}{\sigma}\right]^2\right)}\,O\left[\frac{1}{\sigma}\right]^3\Big\}\Big\}$$

We also test the formal solution by verifying that term cancel in powers of $i-1$ th of σ in the differential equation for $i = 1, 2$ which involve the solved terms.

In[250]:=
```
TestLHS1 = D[P1, t] + P1.D[Q1, t] + P2.A0;
TestRHS1 = A1.P1 + A0.P2 + A2;
Testeqn1 = Simplify[TestLHS1 - TestRHS1]
```

Out[252]= {{0, 0}, {0, 0}}

In[253]:=
```
P3 = Pmtc[3];
TestLHS2 = Ondiag[D[P2, t] + P2.D[Q1, t] + P3.D[Q0, t]];
TestRHS2 = Ondiag[A1.P2 + A0.P3 + A2.P1];
Simplify[TestLHS2 - TestRHS2]
```

Out[256]= {{0, 0}, {0, 0}}

We obtain our asymptotic estimate $P\mathcal{F}$ for the original system and list the corresponding exponential terms along with correction terms PP_i:

In[257]:=
```
Asympt = P.Formal;
```

In[258]:=
```
Coefficient[Asympt, σ, 0]
Coefficient[Asympt, σ, -1] / Coefficient[Asympt, σ, 0];
Coefficient[Asympt, σ, -2] / Coefficient[Asympt, σ, 0];
```

Out[258]= $\left\{\left\{-\frac{i\, e^{-i\sigma\int\frac{1}{c[t]}dt}\, t\,\sqrt{\rho[t]}}{\sqrt{c[t]}},\ \frac{i\, e^{i\sigma\int\frac{1}{c[t]}dt}\, t\,\sqrt{\rho[t]}}{\sqrt{c[t]}}\right\},\ \left\{\frac{e^{-i\sigma\int\frac{1}{c[t]}dt}\,\sqrt{c[t]}}{t\,\sqrt{\rho[t]}},\ \frac{e^{i\sigma\int\frac{1}{c[t]}dt}\,\sqrt{c[t]}}{t\,\sqrt{\rho[t]}}\right\}\right\}$