# Computational Local Solvability of Partial Differential Operators: Decidability and numerical solutions involving operators on $H_1$.

C. Winfield

## 1 Introduction and General Theory

We will study the local solvability of a specific class of partial differential operators. We study those which are left-invariant on the Heisenberg group $\mathbb{H}_1$ (i.e. invariant under group action). Our analysis will involve the study of solutions to certain ordinary differential equations derived from a certain representation of these operators. Before we describe in detail the class of operators and methods in question, let us review the definition of local solvability and make precise what classes of functions we will consider for the proposed research. A partial differential operator with smooth coefficients (not all zero) expressed in multi-index operator notation

$$L = \sum_{|\alpha|=0}^{n} a_\alpha(\vec{x})\partial_{\vec{x}}^{\alpha} \tag{1.1}$$

and defined for $\vec{x}_0 \in \mathbb{R}^n$ will be said to be ($C^\infty$-) locally solvable if the following hold:

**Definition 1.1** *For every $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ there is a $u \in \mathcal{C}^\infty(\mathbb{R}^n)$ and a neighborhood $\Omega \ni \overrightarrow{x}_0$ so that $Lu = f$ holds on $\Omega$, perhaps in the sense of distributions. Moreover, $L$ is locally solvable if it is locally solvable at every $\vec{x}_0$*

We note that no conditions are placed on the form of boundary/initial conditions are not specified in the above definition. Some common examples of operators which are locally solvable include all such ordinary differential operators and all such partial differential operators with constant coefficients. Various other classes of operators have been studied in the context of local (non-) solvability, including operators of principle type [NT1, NT2, Hö2, BF], certain hyperbolic operators [N], certain transversally elliptic operators [CR], certain classes of second-order operators [FS, GT], and some operators defined (left-invariant) on the Heisenberg group, second-order or higher [M1, M2, M3, MR].

A well-known example of an operator that is not locally solvable is given by H. Lewy [L], defined on $\mathbb{R}^3$ can be (applying a change of variables) written as

$$L_{\text{Lewy}} = \partial_x + i(\partial_y + x\partial_t)$$

The class of operators we study is related to this operator involving the vector fields $X = \partial_x$ and $Y = \partial_y + x\partial_t$. We find these to be left-invariant on $\mathbb{H}_1$ as we show as follows: Denote by $\mathcal{T}_{\vec{x}'} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ the group action on $\mathbb{R}^3$ given by

$$\mathcal{T}_{\vec{x}'}(\vec{x}) = (x + x', y + y', t + t' + x'y)$$

we find that for $f \in C^\infty(\mathbb{R}^3)$

$$V_{\vec{x}}(f(\mathcal{T}_{\vec{x}'}(\vec{x})) = V_{\mathcal{T}_{\vec{x}}(\vec{x})} \circ f(\mathcal{T}_{\vec{x}'}(\vec{x}))$$

for the vector field $V_{\vec{x}} = X$ or $Y$. Although it is possible to solve the equation $Lu = f$ for some functions $f$, we conclude that the equation cannot be solved for some smooth $f$. For instance, we see [F] that $L_{Lewy}u = f$ is not solvable in any neighborhood of the origin when $f$ is $C^\infty(\mathbb{R}^3)$ but not analytic. Some such attempts using Mathematica differential equation solvers yield only general homogeneous solutions.

The proposed research will be a study of the local (non-) solvability of the following operators which, via operator notation, can be uniquely express by polynomials $P$ in two general non-commuting variables $X = \partial_x$ and $Y = \partial_y + x\partial_t$. We restrict our class of polynomials $P$ as follows:

1) $P$ is of degree $n \geq 2$.

2) $P$ is homogeneous in the general variables $X$ and $Y$.

3) In the complex variable $z$, the polynomial $P(iz, 1)$ has distinct roots $\{\gamma_j\}_{j=1}^n$ with distinct real parts.

4) The complex polynomial $P(iz, 0) = z^n$.

We therefore study operators which can be written in the form

$$L = (-i)^n X^n + \sum_{j=1}^{n} a_j Y^j X^{n-j} + E(X, Y)$$

all with complex constant coefficients $a_j$ and where $E$ is a finite, linear combination of terms of order $n$ all involving at least one commutator factor $[X, Y]$. Property 4 assures the presence of the term $X^n$ and is convenient for our analysis which follows [C]. Property 3 may be generally be relaxed, allowing merely distinct $\gamma_j$'s, but the above suffices for our proposal. For the rest of the proposal we will call these polynomials *generic*.

Most of our development will rely on results of [W2] which exploits asymptotic methods of [C]. Here a certain representation is analyzed by producing asymptotic estimates which depend on the $\gamma_j$ and various coefficient of $P$. In particular, we study solutions to $\mathcal{L}^\pm y(u) = 0$ where

$$\mathcal{L}^\pm = P(\mp i\partial_u, -u).$$

The set of such solutions to $\mathcal{L}y = 0$ is called the *kernel*, denoted by $\ker \mathcal{L}$. We denote by superscript $*$ the adjoint of an operator. We now state the main theorem which we use to determine solvability.

**Theorem 1.2** *Let $P$ be a generic polynomial. Then, the operator $L = P(X,Y)$ is locally solvable if and only if the kernels of $(\mathcal{L}^{\pm})^*$ contains no Schwartz-class (on $\mathbb{R}^1$) functions other than the zero function.*

Recall that Schwartz class functions $\mathcal{S}(\mathbb{R}^n)$ are those smooth functions $f$ for which

$$(1+|\vec{x}|)^a \partial_{\vec{x}}^{\beta} f(\vec{x})$$

is bounded on $\mathbb{R}^n$ for every real $a$ and multi-index $\beta$.

For $\varphi \in \mathcal{S}(\mathbb{R}^3)$, using operator notation, we write

$$P(X,Y)\varphi(x,y,w) =$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-i(y\xi+w\eta)} P(\partial_x, -i(\xi+\eta x))\hat{\varphi}(x,\xi,\eta)\, d\xi\, d\eta.$$

Here we define

$$\hat{\varphi}(x,\xi,\eta) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \varphi(x,y,w)e^{i(y\xi+w\eta)}\, dy\, dw$$

and

$$\check{\varphi}(x,\xi,\eta) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \varphi(x,y,w)e^{-i(y\xi+w\eta)}\, dy\, dw.$$

The operators $\mathcal{L}^{\pm}$, as defined above, are related to the operator $P(\partial_x, -i(\xi+\eta x))$ by a change of variables as follows: Set

$$u^{\pm}(x,\eta,\xi) = x\sqrt{\eta} \pm \xi/\sqrt{\eta} \text{ (resp.)}.$$

For $\varphi \in C^n(\mathbb{R})$,

$$\begin{aligned} P(\partial_x, -i(\xi+\eta x))\varphi(u^{\pm}) &= P(\partial_x, -i|\eta|^{1/2}(\xi|\eta|^{-1/2} \pm x|\eta|^{1/2})\varphi(u^{\pm}) \qquad (1.2)\\ &= (i)^n P(-i\partial_x, \pm|\eta|^{1/2}(\mp\frac{\xi}{|\eta|^{1/2}} - x|\eta|^{1/2}))\varphi(u^{\pm})\\ &= (i)^n|\eta|^{n/2}P(-i\partial_u, \mp u)\varphi(u)\\ &= (i)^n|\eta|^{n/2}(\mathcal{L}^{\pm}\varphi)(u^{\pm}) \end{aligned}$$

for $\eta \gtrless 0$ (resp.). According the sign changes defining $\mathcal{L}^{\pm}$ the characteristic roots are given by $\{\pm\gamma_j\}_{j=1}^n$, respectively.

We will order the characteristic roots so that $\mathrm{Re}\gamma_j < \mathrm{Re}\gamma_{j+1}$ for each $1 \leq j < n$ and describe canonical bases according to asymptotics from [C]. Since $\mathcal{L}^{\pm}$ are of the general class form, we will simply denote the ordinary operators by $\mathcal{L}$ and use the $\pm$ superscript in association with estimates on $\mathbb{R}^{\pm}$

**Lemma 1.3** *Let $\mathcal{L} = P(-i\partial_x, -x)$ for some generic $P$ with characteristic roots $\gamma_j$ as above. Then there are functions $\phi_j^{\pm}(x) \in \mathcal{C}^{\infty}(\mathbb{R})$ defined on $\mathbb{R}\times\mathbb{C}$, with the following properties:*

*i) The sets* $\{\phi_k^+(x)\}_{k=1}^n$ *and* $\{\phi_k^-(x)\}_{k=1}^n$*, form bases of* $\ker\mathcal{L}$
*ii) for some complex exponents,* $\rho_k(P)$*, for* $0 \le j \le n-1$*,* $1 \le k \le n$*.*

$$\frac{d^j}{d\,x^j}\phi_k^\pm(x) = e^{-\gamma_k x^2/2}((-\gamma_k)^j x^{\rho_k+j} + o(1)) \tag{1.3}$$

*as* $x \to \pm\infty$*, respectively;*

Here we take $x^\rho = e^{\log\rho}$ with a branch of complex log including the positive $+x$-axis and with the argument take to be $i\pi$ along the $-x$-axis.

Let us introduce some notation: The expression $f \lesssim g$ means that $f \le Cg$ for positive constant, $C$, independent of $x$ and $z$ and $f \asymp g$ means that the ratio $f/g$ is bounded above and below by finite, positive constants, independent of $x$. Our method in using Theorem 1.2 involves a study of global behavior of functions in $\ker\mathcal{L}$ for those operators arising in (1.2). We consider classes of ordered bases $\phi_k^\pm$ which satisfy

$$\frac{d^j}{d\,x^j}\phi_k^\pm \lesssim (1+|x|)^{a+j}e^{-Re\gamma_k x/2} \tag{1.4}$$

for $x \gtrless 0$ for some real constant $a$. Such pairs of bases $\{\{\phi_k^+\}_{k=1}^n, \{\phi_k^-\}_{k=1}^n\}$ will be called *admissible*. Here,

$$\lim_{x\to\pm\infty}\frac{\phi_{j+1}^\pm(x)}{\phi_j^\pm(x)} = 0 \tag{1.5}$$

(resp.) for each $1 \le j < n$ and choice of $\pm$ sign.

For a given operator $\mathcal{L}$, admissible bases of $\ker\mathcal{L}$ have the following properties:

1) Given any pair of admissible bases, there is an invertible matrix $A$ so that

$$\begin{pmatrix}\phi_1^-\\ \vdots\\ \phi_n^-\end{pmatrix} = A\begin{pmatrix}\phi_1^+\\ \vdots\\ \phi_n^+\end{pmatrix}.$$

2) Given a pair of admissible bases $\{\{\phi_k^+\}_{k=1}^n, \{\phi_k^-\}_{k=1}^n\}$ and constant, $n\times n$, invertible, upper-triangular matrices $A_1$ and $A_2$, the bases $\{\{\tilde\phi_k^+\}_{k=1}^n, \{\tilde\phi_k^-\}_{k=1}^n\}$ resulting from

$$\begin{pmatrix}\tilde\phi_1^-\\ \vdots\\ \tilde\phi_n^-\end{pmatrix} = A_1\begin{pmatrix}\phi_1^-\\ \vdots\\ \phi_n^-\end{pmatrix}.$$

$$\begin{pmatrix}\tilde\phi_1^+\\ \vdots\\ \tilde\phi_n^+\end{pmatrix} = A_2\begin{pmatrix}\phi_1^+\\ \vdots\\ \phi_n^+\end{pmatrix}$$

also form an admissible pair.

3) For a permutation $\sigma$ (on $n$ letters) denote by $I_\sigma$ the matrix with zero elements except for (leading) ones in the positions $(\sigma(j), j) : 1 \leq j \leq n$. Then, there is a unique $\sigma$ such that for some admissible pair of bases $\{\{\psi_k^+\}_{k=1}^n, \{\psi_k^-\}_{k=1}^n\}$ we obtain

$$\begin{pmatrix} \psi_1^- \\ \vdots \\ \psi_n^- \end{pmatrix} = I_\sigma \begin{pmatrix} \psi_1^+ \\ \vdots \\ \psi_n^+ \end{pmatrix}.$$

We note that on the class of invertible $n \times n$ matrices, left and right multiplication by upper-triangular matrices induces an equivalence class of matrices: For every matrix $A$ there are upper-triangular matrices $U$ and $V$ so that $UAV = I_\sigma$ for some associated permutation $\sigma$. Letting $0 \leq J \leq n$ be the least such index that $\text{Re}{-\gamma_j} > 0 \ \forall\ j \geq J$, we state the following

**Definition 1.4** *A permutation associated with $I_\sigma$ will be called resolving if the following holds $\forall j$ :*

$$\sigma(j) < J \implies j \geq J$$

We apply the characterizations to operators $\mathcal{L}^*$: Since these operators are also of associated with some generic $P$, their kernel spaces have canonical bases following the asymptotic estimates as in Lemma 1.3 but with characteristic roots $\{-\overline{\gamma_j}\}_{j=1}^n$ so that (after rearrangement) there is an associated admissible pair of bases of $\mathcal{L}^*$ satisfying

$$\phi_j^\pm(x) \lesssim e^{+Re\gamma_j x^2/2}(1 + |x|)^b$$

for $x \gtrless 0$, respectively, for some real $b$. where are then able to characterize the property of local solvability of operators $L$ according to equivalence classes of transition matrices $A$ where

$$\begin{pmatrix} \phi_1^- \\ \vdots \\ \phi_n^- \end{pmatrix} = I_\sigma \begin{pmatrix} \phi_1^+ \\ \vdots \\ \phi_n^+ \end{pmatrix}$$

for the same $\sigma$ associated with $\mathcal{L}$. Hence, $\ker \mathcal{L}^*$ has a non-trivial $\mathcal{S}(\mathbb{R})$ function if and only if the permutation $\sigma$ associated with $\mathcal{L}$ is not resolving.

This fact leads to the following reformulation of Theorem 1.2.

**Proposition 1.5** *The operator $L = P(X, Y)$ for a generic $P$ is locally solvable if an only if the permutations $\sigma^\pm$ associated to each of* $\ker \mathcal{L}^\pm$ *(resp.) are both resolving.*

Retaining the superscript $\mathcal{L}^\pm$ of Theorem 1.2, we note some cases for which the determination of local solvability is immediate.

**Corollary 1.6** *The operator $L = P(X, Y)$ is locally solvable if the characteristic roots $\gamma_j$ satisfy $Re\gamma_j = 0 \ \forall j$.*

**Corollary 1.7** *The operator $L = P(X,Y)$ in not locally solvable if one of the following holds:*

*1* $Re\gamma_j > 0 \forall j$.

*2* $Re\gamma_j < 0 \forall j$.

The approach of our proposed project to this class of differential equations can be viewed in four major computational steps:

1) We compute admissible pairs of bases for each of the associated $\mathcal{L}^{\pm}$, ordered according to their asymptotic growth as $x \to \pm\infty$.

2) We compute the associated transition matrices $A$ for the admissible pairs.

3) We test the general ($C^{\infty}$-) solvability of an operator $P(X,Y)$ according to the equivalence classes of the transition matrices as in Proposition 1.5.

4) Where the test shows that $L$ is locally solvable, we provide a method of solution - at least in a neighborhood.

With concrete methods of testing and calculation at hand in the literature, each of these stages lend themselves to computer program development; either in automation or in well-defined routines that users can follow using standard techniques. More detailed outlines of such procedures are offered in the various sections below.

# Asymptotic Estimates

In order to classify and work with ordered bases $\{\phi_j^{\pm}\}_{j=1}^n$ of solutions to $\mathcal{L}y = 0$, we will need a method to compute asymptotic estimates as in Lemma 1.3 apart from calculating numerical/analytical solutions. To do this, we follow the method as in [C]: Let $\mathcal{L}$ be an operator given by

$$\mathcal{L} = \partial_x^n + \sum_{j=0}^{n-1} x^{n-j} q_j(x) \partial_x^j$$

where the $q_j$'s are rational functions of the form $q_j = d_j + e_j x^{-2} + O(x^{-4})$ for constants $d_j, e_j$. We find the characteristic roots $\gamma_j$ by finding the roots of

$$z^n + \sum_{j=0}^{n-1} (-1)^{n-j} d_j z^j = 0. \tag{1.6}$$

Now to find the exponents $\rho_j$, we set

$$S_0 = \begin{pmatrix} 1 & 1 & \dots & 1 \\ -\gamma_1 x & -\gamma_2 x & \dots & -\gamma_n x \\ \vdots & \vdots & & \vdots \\ (-\gamma_1 x)^{n-1} & (-\gamma_2 x)^{n-1} & \dots & (-\gamma_n x)^{n-1} \end{pmatrix}$$

$$\mathcal{E}_1 = \frac{1}{x^2} \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ (-x)^n e_0 & \dots & (-x)^1 e_{n-1} \end{pmatrix}$$

Then, these exponents are given by

$$\rho_j = x[S_0^{-1}\mathcal{E}_1 S_0 - S_0^{-1} S_0]_{j,j} \tag{1.7}$$

where $[\ ]_{j,j}$ denotes the $j$-th diagonal element and $'$ denote derivative w.r.t $x$.

As an example, let us consider second-order ($n = 2$) operator of this class

$$\mathcal{L} = \partial_x^2 + ax\partial_x + x^2(b + c/x^2)$$

for real (say) $a, b$ with $a^2 > 4b$. We compute $\gamma_{1,2} = \frac{a \pm \sqrt{a^2 - 4b}}{2}$ (resp.)

$$S_0 = \begin{pmatrix} 1 & 1 \\ -\gamma_1 x & -\gamma_2 x \end{pmatrix}, S_0' = \begin{pmatrix} 0 & 0 \\ -\gamma_1 & -\gamma_2 \end{pmatrix}, \mathcal{E}_1 = \begin{pmatrix} 0 & 0 \\ -c & 0 \end{pmatrix},$$

$$S_0^{-1} = \frac{1}{x(\gamma_1 - \gamma_2)} \begin{pmatrix} -\gamma_2 x & -1 \\ \gamma_1 x & 1 \end{pmatrix}$$

$$\rho_1 = \frac{\gamma_1 - c}{\gamma_2 - \gamma_1}, \quad \rho_2 = \frac{-\gamma_2 + c}{\gamma_2 - \gamma_1}$$

with $\gamma_1 < \gamma_2$. One expects similar results [W2] where we allow for distinct roots $\gamma_j$ with some with equal real parts.

We seek to automate the above procedure for such operators and we expect to complete such work within the first six months of the project. We will provide algorithms where a user can identity an operator by coefficients $a, b, c$ whereby characteristic roots will automatically calculated. We will further develop means with which to automatically identify and determine local solvability in those cases where local solvability can immediately be determined - such as those as in the cases of Corollaries 1.6 and 1.7 for instance. Similar procedures will be developed for higher-order cases $n > 2$. Although the computations above are among the initials steps of the proposed project, these procedures alone will lead to publications on asymptotic methods of ordinary differential equations accessible to graduate and undergraduate student in subject areas involving applications of analysis. In particular, tutorial exercises will be developed for students to compare solutions of related ordinary differential equations to those asymptotic estimates $e^{-\gamma_j x^2/2} x^{\rho_j}$.

## Characterizations of Bases: Case $n = 2$

As a demonstration of our proposed methods we apply our test to second-order differential operators whose local (non-) solvability is known. We will examine the case

$$L = -X^2 - Y^2 + i\lambda[X, Y] \tag{1.8}$$

for constant $\lambda$. This case has been studied for constant $\lambda$ [W2] and non-constant $\lambda = \lambda(x)$ cases [CK, CKM]. The associated ordinary differential operators are given by

$$\mathcal{L}^{\pm} = \partial_x^2 - x^2 \mp \lambda \text{ (resp.).} \tag{1.9}$$

We have from the well-known Hermite differential equation [CL] that $\ker \mathcal{L}^{\pm}$ contains functions of class $\mathcal{S}(\mathbb{R})$ if and only if $\mp\lambda = 2\mathfrak{l} + 1$ (resp.) for some $\mathfrak{l} \in \mathbb{N}$. Therefore, $L$ is locally solvable if and only if $\pm\lambda \neq 2\mathfrak{l} + 1$ for any such $\mathfrak{l}$.

Let us demonstrate a more naive approach in order to demonstrate out computational methods. We see that Mathematica software provides a general solution to the equation

$$y''(x) - x^2 y(x) = \lambda y(x)$$

Let us restrict our case for now to that of $\lambda \geq 0$. The general solution is given in terms of parabolic cylinder functions by [AS] (see equations 19.4.3, 19.8.1, 19.8.2)

$$y = C_1 V(\lambda/2, x) + C_2 U(\lambda/2, x) \tag{1.10}$$

where

$$\pi V(a, x) = \Gamma(\frac{1}{2} + a)[\sin(\pi a) U(a, x) + U(a, -x)]$$

We note that in this case $-\gamma_{1,2} = +1, -1$ and $\rho_{1,2} = \frac{-1+\lambda}{2}, -\frac{1+\lambda}{2}$ (resp.) We naively test the bases functions $U, V$ for in order to demonstrate how to construct admissible bases: Consider the following limits (which can be computed by Mathematica)

$$\begin{aligned} &\lim_{x\to+\infty} \frac{U(\lambda/2, \sqrt{2}x)}{e^{x^2/2} x^{\rho_1}} = 0 && \lim_{x\to+\infty} \frac{U(\lambda/2, \sqrt{2}x)}{e^{-x^2/2} x^{\rho_2}} = 1 \\ &\lim_{x\to+\infty} \frac{V(\lambda/2, \sqrt{2}x)}{e^{x^2/2} x^{\rho_1}} = \sqrt{\frac{2}{\pi}} && \lim_{x\to+\infty} \frac{V(\lambda/2, \sqrt{2}x)}{e^{-x^2/2} x^{\rho_2}} = +\infty \end{aligned} \tag{1.11}$$

Such a test as above can be implemented by Mathematica

At this point we see from the limits above and verify by [AS] that

$$\begin{aligned} V(\lambda/2, \sqrt{2}x) &\asymp e^{x^2/2} x^{\rho_1} \\ U(\lambda/2, \sqrt{2}x) &\asymp e^{-x^2/2} x^{\rho_2} \end{aligned}$$

for sufficiently large $x > 0$, so that we can assign

$$\phi_1^+(x) = V(\lambda/2, \sqrt{2}x), \;\; \phi_2^+(x) = U(\lambda/2, \sqrt{2}x)$$

In order to complete our construction we need to obtain a basis with such distinct estimates for large $x < 0$. We note that from 19.8.3 [AS] and (1.11)

$$\lim_{x\to-\infty} \frac{\phi_1^+(x)}{\phi_1^+(-x)} = \sin(\lambda\pi/2), \qquad \lim_{x\to-\infty} \frac{\phi_1^+(x)}{e^{x^2/2}|x|^{\rho_1}} = \sqrt{\frac{2}{\pi}}\sin(\lambda\pi/2), \tag{1.12}$$

$$\lim_{x\to-\infty} \frac{\phi_2^+(x)}{\phi_1^+(-x)} = \frac{\pi}{\Gamma(\frac{\lambda+2}{2})}, \qquad \lim_{x\to-\infty} \frac{\phi_2^+(x)}{e^{-x^2/2}|x|^{\rho_2}} = \frac{\pi}{\Gamma(\frac{\lambda+2}{2})}$$

For $\pm\lambda$ not an odd integer, let us choose

$$\phi_1^-(x) = \phi_2^+(x)$$

and set

$$\phi_2^- = C\phi_1^+ + \phi_2^+$$

and choose (uniquely) $C$ so that $\phi_2^-(x) \asymp e^{-x^2/2}|x|^{\rho_2}$ as for large $x < 0$. The existence of such a $C$ is assured by Theorem 1.2. From the limits above, it is not difficult to show that

$$C = -\frac{\pi\sin(\pi\lambda/2)}{\Gamma(\frac{\lambda+1}{2})} \stackrel{\text{def}}{=} C_\lambda$$

$$\begin{pmatrix}\phi_1^- \\ \phi_2^-\end{pmatrix} = \begin{pmatrix}1 & 0 \\ C_\lambda & 1\end{pmatrix}\begin{pmatrix}\phi_1^+ \\ \phi_2^+\end{pmatrix} \tag{1.13}$$

Here it is not difficult to show that the transition matrix $A$ satisfies

$$A \sim \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix} \tag{1.14}$$

If $\lambda$ is an even integer, we may choose

$$\phi_1^-(x) = \phi_2^+(x), \;\; \phi_2^- = \phi_1^+(x)$$

so that

$$\begin{pmatrix}\phi_1^- \\ \phi_2^-\end{pmatrix} = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}\phi_1^+ \\ \phi_2^+\end{pmatrix} \tag{1.15}$$

and the desired property of the transition matrix is clear.

If $-\lambda = 2m + 1$ is an odd natural number, then we find that $\phi_2^+(x) = (-1)^m\phi_2(-x)$ so that by choosing $\phi_2^-(x) = \phi_2^+(x)$, we may conclude that

$$\lim_{x\to-\infty} \frac{\phi_2^+(x)}{e^{x^2/2}|x|^{\rho_1}} \neq 0$$

Therefore, the choice $\phi_j^-(x) = \phi_j^+(x) : j = 1, 2$ forms an admissible pair whose associated transition matrix $A$ is the identity matrix, not satisfying (1.14). In summary, we conclude that the transition matrices $A$ for operator $\mathcal{L}$ satisfy (1.14) if and only if $-\lambda$ is not a odd natural number.

We note the operator $\mathcal{L}^-$ is identical to $\mathcal{L}^+$, but with $\lambda$ replaced by $-\lambda$. Then it is clear that the associated matrices $A$ satisfy (1.14) if and if $+\lambda$ is not an odd natural number. Therefore, the operator $P(X,Y)$ is locally solvable if and only if $\lambda$ is not an odd integer (cf. [CKM, W2]). Similar analysis applied for general generic second-order operators as the resulting bases can be expressed by various parabolic cylinder functions.

With analytical justification established for our operators of Hermite type, let us outline a more general computational procedure to produce admissible bases. From this we form computer algorithms. Let us review the case $n = 2$ : Given a basis of solutions $\psi_1(x)$, $\psi_2(x)$ to $\mathcal{L}y = 0$ and compute

$$c^{\pm}_{j,k} = \lim_{x\to\pm\infty} \frac{\psi_j(x)}{e^{\gamma_j x^2/2}|x|^{\rho_j}}$$

(resp.). Then, choose $l^{\pm}$ so that $c^{\pm}_{l^{\pm},1} \stackrel{\text{def}}{=} C^{\pm}_1$ (resp.) is finite and non-zero and set $\phi^{\pm}_1(x) = \psi_{l^{\pm}}(x)$ (resp.).

Then, in order produce $\phi^{\pm}_2(x)$ with the desired asymptotic growth, let $m^{\pm}$ be the indices not equal $l^{\pm}$; here, $m^{\pm} = 3 - l^{\pm}$ (resp.). Then, we set $C^{\pm}_2 = c^{\pm}_{m^{\pm},2}$

$$\begin{aligned} \phi^{\pm}_2(x) &= \phi_1(x) - \frac{C^{\pm}_1}{C^{\pm}_2}\psi_{m^{\pm}}(x) \\ &\stackrel{\text{def}}{=} \psi_{l^{\pm}}(x) - \frac{C^{\pm}_1}{C^{\pm}_2}\psi_{m^{\pm}}(x) \end{aligned} \tag{1.16}$$

We see that there are invertible matrices $A^{\pm}$ so that

$$\begin{pmatrix} \phi^{\pm}_1 \\ \phi^{\pm}_2 \end{pmatrix} = A^{\pm} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

(resp.) so that we obtain an admissible pair of bases with

$$\begin{pmatrix} \phi^{-}_1 \\ \phi^{-}_2 \end{pmatrix} = A^{-}(A^{+})^{-1} \begin{pmatrix} \phi^{+}_1 \\ \phi^{+}_2 \end{pmatrix}$$

whose transition matrix is thus given by $A = A^{-}(A^{+})^{-1}$. With the matrix $A$ in hand, we check the element $[A]_{2,1}$. As we see above, if $[A]_{2,1} \neq 0$, then $A \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. If $[A]_{2,1} = 0$, then $A \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Results here can be compared to already known in the literature [CK, CKM, MR].

In the course of providing computational schemes for the above procedure to perform on Mathematica, we will have opportunities to develop student-centered documents and exercises serving as introduction to global analysis. Moreover, we will have been able to train assistants to examine many examples in preparation for higher-order operators.

## Higher-Order cases $n \geq 2$

As we work on higher-order operators $P(X, Y)$ of degree $n$, we are faced with classifying larger transition matrices $A$ to determine solvability. Supposing that we can construct these matrices with sufficient accuracy, we can follow a method [W2] (see Proposition 3.4 therein) to determine their equivalences classes in the sense of Proposition 1.5. Indeed, we see that one can reduce any invertible matrix to one of the form $I_\sigma$ by way of certain elementary row operations. Moreover, since the proof is argued by induction, it is clear that we can develop a procedure that is easy for the user to follow - perhaps an automated procedure. With such software developed we will try a large number of operators and report our discoveries of locally and non-locally solvable operators as we have project members various subclasses of our operators. We anticipate a large number of results to be documented in an encyclopedic text made available for publication. Such results are expected between the eighteenth and twenty-forth months of the project.

For an operator of the form $\mathcal{L}$ we can produce a basis of $\ker \mathcal{L}$ by solving (symbolically or numerically) for functions $\psi_j(x) : j = 1, \ldots, n$ such that the Wronskian is non-zero. For instance we may specify that

$$\mathcal{L}\psi_j = 0, \ \ \psi^{(k)}(0) = 0 : 0 \leq k \leq n-1 \,\&\, k \neq j, \ \ \psi^{(j-1)}(0) = 1 \tag{1.17}$$

Here, the Wronskian $W = W(\psi_1, \ldots, \phi_n)(x)$ satisfies

$$W = \begin{pmatrix} \psi_1(x) & \psi_2(x) & \ldots & \psi_n(x) \\ \frac{d}{d\,x}\psi_1(x) & \frac{d}{d\,x}\psi_2(x) & \ldots & \frac{d}{d\,x}\psi_n(x) \\ \vdots & \vdots & & \vdots \\ \frac{d^{n-1}}{d\,x^{n-1}}\psi_1(x) & \frac{d^{n-1}}{d\,x^{n-1}}\psi_2(x) & \ldots & \frac{d^{n-1}}{d\,x^{n-1}}\psi_n(x) \end{pmatrix} \tag{1.18}$$

Here $W(0) = 1$ and, hence, by Abel's formula we see that (cf. [W2, C])

$$W(x) = W(0)e^{-\gamma x^2/2} = e^{-\gamma x^2/2}$$

for $\gamma \stackrel{\text{def}}{=} \sum_{j=1}^n \gamma_j$. Here we may estimate $c^{\pm}_{k,j}$ numerically by estimating the various ratios

$$c^{\pm}_{k,j} = \frac{\psi_k(x)}{e^{-\gamma_j x^2/2}|x|^{\rho_j}}$$

for large negative and positive $x$. Thus, we guarantee the (theoretical) existence of a bases suitable for the entire real line. We will compare numerical solutions with those derived from alternate schemes. For instance, we will investigate solutions with initial conditions according to asymptotic behavior.

$$\frac{d^k}{d\,x^k}\phi^{\pm}_j(x) = (\gamma_j x)^{k-1} e^{-\gamma_j x^2/2}|x|^{\rho_j}$$

at large $x_0 > 0$ (say). We will see if there are any advantages to this scheme in terms of cost of computation or in error propagation as we compute transition matrices $A$.

With at least one basis at hand, we do, however, expect errors to propagate in the numerical results over large intervals starting at $x = 0$. These errors will effect our calculations of the transition matrices $A$. We do nonetheless expect that results are tractable in identifying solvability for a large class of operators $P(X, Y)$. For instance, in the case $n = 2$ we expect to find the value of $[A]_{2,1}$ to high precision which can be estimated. We can therefore be certain of local solvability once $[A]_{2,1}$ is known to be non-zero. In cases were $[A]_{2,1}$ is small, we expect to analyze and report some measure of certainty of our calculations in determining non-solvability. For higher $n$ we know that a certain number of linear operations on the rows of $A$ are involved. Hence, we can likewise be certain of our conclusions for many such cases where $P(X, Y)$ is locally solvable - or otherwise at least still be able to give a measure of accuracy of our conclusions.

## Further Implementation: Calculation of Solution and Exploration of Related Operators

Let us take a glimpse into possible several avenues our research program can take beyond determination of ($C^\infty$) local solvability. Once an operator $P(X, Y)$ is determined to be the proposed research take the methods of [W2] to construct solutions to $LH = f$. We have that $H$ is a finite sum of smooth functions, each of which are partial Fourier transforms of functions which can be written in the form

$$\sum_{j=1}^{n} \phi_j(u) \int_{c_j}^{u} \frac{W_j(\tau)}{W(\tau)} g(x, \xi, \eta, \tau) d\tau : \tag{1.19}$$

Here, $u = x\sqrt{\eta} + \xi/\sqrt{\eta}$ for Fourier variables $\eta, \xi > 0$; the function $g$ is bounded and continuous (and decaying sufficiently rapidly as $\xi, \eta \to \infty$) which depends on $f$ and the neighborhood of solution; the functions $\phi_j$ form and admissible basis of $ker\mathcal{L}$; the function $W_j(x)$ is given by a constant times the determinant

$$\begin{array}{c} \begin{array}{ccccc} & & j\!-\!th\ column & & \end{array} \\ \begin{vmatrix} \phi_1 & \cdots & 0 & \cdots & \phi_n \\ \frac{d}{dx}\phi_1 & \cdots & 0 & \cdots & \frac{d}{dx}\phi_n \\ \vdots & \cdots & \vdots & \cdots & \\ \frac{d^{n-1}}{dx^{n-1}}\phi_1 & \cdots & 1 & \cdots & \frac{d^{n-1}}{dx^{n-1}}\phi_n \end{vmatrix} \end{array}$$

(formed by replacing the $j$th column of $W(x)$ by the transpose of $(0, \ldots, 0, 1)$); and, the limits $c_j$ are finite or infinite, depending on the behavior of $W_j(x)/W(x)$ for large $x$. This formula results by following the variation of parameters formula [CL]) where the $c_j$ are determined by the associated transition matrix $A$. What's more, the solution $H$ can be constructed on any bounded neighborhood of $R^3$.

Differential equation solvers such as those used in Mathematica do not provide solutions for some $n = 2$ cases of $Lf = g$ with analytic $g$ although $L$ may not be $\mathcal{C}^\infty$)-locally solvable (eg. $-X^2 + Y^2 = \sin(x + y)$ yields no solution when using the 'DSolve' package of Mathematica). With computational algorithm arising from our project, such published work will add to the public literature of numerical solutions to differential equations. Related publications will serve as tutorials in performing computations using the general method of variations of parameters to be accessible to students and laypeople interested in ordinary differential equations. Moreover, we expect to refine our results as we investigate other modes of solvability - other that $C^\infty-$local solvability. We note that even for the operator $L= L_{Lewy}$ the equation $Lf = g$ is solvable for some smooth $g$. Likewise some $Lf = g$ may likely have solution though $L = P(X, Y)$ is not locally solvable. We will make available to users of our software some guidance, where feasible, to classes of admissible functions $g$ and an estimate on the smoothness of solution $f$.

With methods similar to those described above (1.19), we will explore related differential, extending out study to those operators of [W1, W3]. We will study operators of the form $L = P(X, Y)$ where $X = \partial_x$ and $Y = \partial_y + x^m\partial_t$ for integers $m \geq 3$. Here, via partial Fourier transforms we are lead to representations of the form

$$L = P(i\partial_x, z - x^m) \tag{1.20}$$

for a large parameter $z > 0$. Here, we would study transition matrices $A = A(z)$ to determine solvability of $L$ as in [W3], according to the behavior of $A(z)$ for large $z$. Here, the implementation of software will benefit the general study of local solvability so that with automation available, the project will serve to allow a detailed study of the infinite families of matrices $A(z)$ as to estimate their asymptotic behavior as $z \to \infty$. Such investigation is as yet prohibitively cumbersome without any technological aids.

Other advances in the general theory of local solvability should result in explorations related to aforementioned work. These involve operators of the form

$$L = P(X, Y) + Q(X, Y) \tag{1.21}$$

where is generic of order $n \geq 2$ with $Q$ as some polynomial of order strictly less than $n$ ($n \geq 2$). In treating these operators, the team will study representation with amount to ordinary differential operators of the form

$$P(i\partial_x, \mu x) + \sum_{j=0}^{n-1} \epsilon^{n-j} Q_j(\partial_x, \mu x).$$

for certain homogeneous polynomials $Q_j$, with complex, constant coefficients, of order $j$ and with real parameters $\mu, \epsilon$. Cases for $n = 2$ will be compared to those of [MPR] whose techniques-based on Laplace transforms - do not apply for higher-order cases $n \geq 3$. Software development will aid the investigations, likely leading to publishable research in real analysis/partial differential equations as test cases can be readily studied.

In summary the proposed project is expected to advance current investigations of a broad class of operators. It is likely that subclasses of such operators may be discovered according to their solvability as parameters of the operators can be varied as the analysis proceeds with relative ease. Aside from theoretical investigations, the project will also produce benefits in applied/computational mathematics, and in mathematics education. The project will result in documentation of large subclasses of operators according their (non-) solvability, within computational certainty of Mathematica software. Indeed, MAST will produce encyclopedic compilations reporting on operators according to their solvability or non-solvability. The project also will provide, with some automation, means of solutions not yet available in Mathematica. Finally, the project will produce educational tools and tutorials, introducing asymptotic methods of ODEs and techniques of Fourier analysis to laypersons and to students of mathematics. Local solvability will show to be broadly accessible.

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