Local Solvability on H_1: Non-homogeneous Operators

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Outline

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Origins

- Discovery (1957): `The Lewy operator`

$$L_{Lewy} = \partial_x + i(\partial_y + x\partial_w)$$

  is NOT locally solvable.

- An operator L is called ($\mathcal{C}^\infty$-) `locally solvable at` $\vec{x}_0$ ($\in \mathbb{R}^n$) if for every smooth ($\mathcal{C}^\infty(\mathbb{R}^n)$) function f there is a function (or distribution) u so that $Lu = f$ `on some neighborhood of` $\vec{x}_0$.
- We'll say simply that L is `locally solvable` if it is locally solvable `at every` x_0 in $\mathbb{R}^n$.
- Vector fields $X = \partial_x$, $Y = \partial_y + x\partial_z$ are LEFT INVARIANT on $\mathbb{H}_1$. (i.e. $\mathcal{T}_{\vec{x}} \circ V = V \circ \mathcal{T}_{\vec{x}}$ for group translation $\mathcal{T}_{\vec{x}}$ where $V = X$ or Y.)

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Homogeneous operators.

Consider operators in the form

$$L = (-iX)^n + \textit{lower order in } X$$

where the replacement $X \to z$, $Y \to 1$ yields a polynomial (in z) with distinct roots.
More precisely, we set $L = P(X, Y)$, in operator notation, where

- P is a *HOMOGENEOUS* polynomial with complex coefficients in the non-commuting variables X, Y.
- In the complex variable z,

$$p(z) \stackrel{\text{def}}{=} P(iz, 1) = z^n + \textit{lower order}$$

  with $n \geq 2$
- $p(z) \stackrel{\text{def}}{=} P(iz, 1)$ has distinct roots $\gamma_j : j = 1, \ldots, n$.
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Theorem

A generic operator L of order $n \geq 2$ is locally solvable if and only if the corresponding ordinary differential equations

$$P(\pm i\partial_x, x)^* y = 0$$

have no Schwartz-class solutions other than $y \equiv 0$.

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Examples

- For distinct real $\alpha_j : j = 1, 2, \ldots, n$ the operators

$$L = \Pi_{j=1}^{n}(X - \alpha_j Y)$$

  is locally solvable.
- Operators of the form

$$L = X^2 + Y^2 + i\lambda[X, Y]$$

  for constant λ is locally solvable if neither of $\pm\lambda$ is an odd integer.
- Indeed, L above is not locally solvable if either of $\pm\lambda$ is an odd integer. The result follows according to the eigenvalues of the `Hermite` ordinary differential operator

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The Setup.

We now consider operators of the form

$$P(X,Y) = P_n(X,Y) + Q(X,Y)$$

where P_n is generic (of order $n \geq 2$) and Q is of order strictly less than n. Let us set

$$\mathcal{L}_\mu^\pm \stackrel{\text{def}}{=} \mu^{-n} P(\pm i\mu\partial_u, \mu u)$$

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respectively. Think of the all but the highest order terms in X, Y vanishing as $\mu \to \infty$.

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For the operator L above suppose that the (generic) polynomial P_n has characteristic roots γ_j all with non-zero real parts. Then L is locally solvable if $P_n(X, Y)$ is locally solvable.

From [W1] we have immediately

Corollary

The operator L above is locally solvable if ker$(\mathcal{L}^{\pm}_{\infty})^* \bigcap \mathcal{S}(\mathbb{R})$ $= \{0\}$ for both choices of $\pm$ sign.

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Results on non-solvability we now state are as follows:

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L is not locally solvable if, for some choice of $\pm$ sign, the set of parameters $\mu \in \mathbb{R}^+$: $ker(\mathcal{L}_\mu^\pm)^* \bigcap \mathcal{S}(\mathbb{R}) \setminus \{0\} \neq \emptyset$ has a limit point in $\mathbb{R}^+$.

In other words, L is NOT locally solvable if the non-linear eigenvalues of $\mathcal{L}_\mu^\pm$ has an accumulation point for some choice of $\pm$.

Theorem

L is not locally solvable if the cardinality of either $\{\gamma_j | Re\gamma_j > 0\}$ or $\{\gamma_j | Re\gamma_j < 0\}$ is greater than $n/2$.

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The Representation.

$$(Lf)(x,y,w) = \frac{1}{2\pi} \int e^{-i(y\xi + w\eta)} P(\partial_x, -i(\xi + x\eta)) \hat{f}(x,\xi,\eta) d\xi d\eta$$

- Change of variables on $P(\partial_x, -i(\xi + x\eta))$ result in studying of $\mathcal{L}_\mu^\pm = \mu^{-n} P(\mp\mu\partial_u, \mu u)$
- Our main ODOp: In the + case (say) for some homogeneous P_j of degree j

$$\mathcal{L}_\mu = \sum_{j=0}^{n} \frac{1}{\mu^j} P_{n-j}(i\partial_u, u)$$

- NOTE the `singularity at` $\mu = 0$!

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Main Estimates.

- There are bases $\{\psi_k^{\pm}(t,\mu)\}_{k=1}^n$ of $\ker\mathcal{L}_\mu$ of functions $\mathcal{C}^\infty(\mathbb{R})$ as functions of t and holomorphic as functions of Re$\mu > 0$ which for $1 \leq k \leq n$ and $0 \leq j$ satisfy

$$\frac{d^j}{d\,t^j}\psi_k^{\pm}(t,\mu) = (\pm\gamma_k t + \beta_k/\mu)^j e^{\gamma_j t^2/2 \pm \beta_k t/\mu}(1 + o(1))$$

as $t \to \pm\infty$ (resp.)

- The β_j's depend on the γ_j's and the coefficients of P_{n-1}.
- Roughly: These estimates can be extended to complex t, on sectors depending on characteristic roots γ_j.
- A key to broad characterization of solvability lies in the study of *transition* matrices $A(\mu)$ where bases $\vec{\psi}^{\pm}$

$$\vec{\psi}^{+} = A(\mu)\vec{\psi}^{-}$$

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Solvability: Divide and Conquer

Solvability is proved by construction (forming a parametrix), by dividing up the the domain of μ to one of `large` $\mu > 0$ and another of μ `on a complex arc` away from 0:

- As we `bypass the singularity at` $\mu = 0$ we apply smooth changes of bases appropriate to various sectors of the complex t plane.
- Solutions to $\mathcal{L}_\mu y = 0$ are manageable for our parametrix since we need only to solve our PDE locally. Our parametrix allows this by applications the famous Theorems of Roche and Cauchy.
- The hypotheses on $P_n(\pm i\partial_x, x)$ render our parametrix manageable for large μ. Again, by locally restricting the solution.

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A `necessary condition` for solvability of PDOp L (Hörmander) is a follows: $\forall \epsilon > 0$, $\exists N > 0$ such that

$$|\int \phi \bar{\Psi}| \leq N ||\phi||_{\mathcal{C}^N} ||L^* \Psi||_{\mathcal{C}^N}$$

for every $\phi, \Psi \in \mathcal{C}^\infty(\mathbb{R}^3)$ supported in $|(x, y, w)| < \epsilon$.
This condition is violated when

- $P_n(X, Y)$ is not locally solvable and $A(\mu_j)$ converges to a limit $A(\infty)$ sufficiently rapidly for some sequence $\mu_j \rightarrow \infty$.
- $P_n(X, Y)$ may or may not be locally solvable but there is a non-trivial $\Psi(x, \mu)$ in ker$\mathcal{L}_\mu$ which is of class $\mathcal{S}(\mathbb{R}) \times \mathcal{C}^\infty(\mathcal{I})$ - for μ on an interval $\mathcal{I}$.
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- $P_n(X, Y)$ is not locally solvable and $A(\mu_j)$ converges to a limit $A(\infty)$ sufficiently rapidly for some sequence $\mu_j \to \infty$.
- $P_n(X, Y)$ may or may not be locally solvable but there is a non-trivial $\Psi(x, \mu)$ in $\ker \mathcal{L}_\mu$ which is of class $\mathcal{S}(\mathbb{R}) \times \mathcal{C}^\infty(\mathcal{I})$ - for μ on an interval $\mathcal{I}$.
- The latter condition my depend on the β_j's which, in turn, depend on the coefficients of P_n and P_{n-1}.

Outline

Second-Order Plus Lower-Order

$$\mathcal{B} \stackrel{\text{def}}{=} -X^2 - ia_1YX + a_2Y^2 - i\alpha[X, Y] - ib_1X + b_2Y + c$$

where a_k, b_k, α, c complex numbers with $a_1^2 \neq 4a_2$.

The operator $L \stackrel{\text{def}}{=} \mathcal{B}^*$ is not locally solvable if one the cases hold:

- $\mathrm{Re}\gamma_1$ and $\mathrm{Re}\gamma_2$ are non-zero and have the same sign;
- $\mathrm{Re}\gamma_1 = 0 > \mathrm{Re}\gamma_2$ and

$$\mathrm{Im}(\gamma_1 - \gamma_2)\,\mathrm{Im}(b_1 + \gamma_1 b_2) > \mathrm{Re}(\gamma_2)\,\mathrm{Re}(b_1 + \gamma_1 b_2); \text{ or}$$

- $\mathrm{Re}\gamma_2 = 0 < \mathrm{Re}\gamma_1$ and

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$$a_2 = \alpha = 0$$

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for some real $\lambda \neq 0$.

- The characteristic roots are 1 and 2λ.
- The associated operator $\mathcal{B}$ is locally solvable when $b_1 = b_2 = c = 0$.
- However, the operator L is not locally solvable for any b_2 and c when $\mathrm{Re}b_1 > 0$, although the associated $P_2(X, Y)$ is locally solvable.

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Generalized Laplacians

Consider for real $-1 < \lambda < 1$ the operator

$$(\lambda^2 - 1)L_{\lambda,\alpha} = (1 - \lambda^2)X^2 + Y^2 + i\lambda(XY + YX) + i\alpha[X, Y]$$

- $L_{\lambda,\alpha}$ is not locally solvable when $\alpha \in \mathbb{H}^+$ is odd;
- yet, for any constant $c \neq 0$, $L_{\lambda,\alpha} + c$ is locally solvable $\forall \lambda, \alpha$.

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Outline

Final Remarks

- For a large subclass of our operators $L = P(X, Y)$ the solvability of the highest order part $P_n(X, Y)$ determine solvability of L.
- However, we have examples where L is locally solvable although $P_n(X, Y)$ is not.
- ....And, vice versa.......
- If we pass the hypotheses on $P_n(\pm i\partial_u, u)$ to conditions on $A^{\pm}(\mu)$ as $\mu \to +\infty$: We produce conditions equivalent to local solvability. (Elaboration here will make the talk too long.)

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