GR Calculations in Specific Bases Using Mathematica

George E. Hrabovsky
MAST

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What I Will Cover

- Introduction
- How to Establish a Manifold
- How to Establish a Coordinate Chart
- How to Define a Metric
- How to Define a Tensor
- Computing the Christoffel Symbols
- The Riemann Tensor, The Ricci Tensor, The Ricci Scalar, and The Einstein Tensor
- The Stress-Energy Tensor
- Einstein's Field Equations

Introduction

This is the third of an apparently endless series of talks on how to use *Mathematica* in general relativity.

Two years ago I talked about the built-in capabilities for handling tensors.

Last year I talked about the xAct package in general and how to apply it to perturbative general relativity, deriving the scalar and tensor field equations for a gravitational perturbation given a Lagrangian.

This year I am talking about performing calculations in specific coordinate bases.

Past talks can be found at the website:

http : // www.madscitech.org/tensors.html

This talk will appear there also.

Establishing Your Manifold

The first thing too do is activate xAct.

```
<< xAct`xTensor`
```

```
------------------------------------------------------------

Package xAct`xPerm`  version 1.2.2, {2014, 9, 28}

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Connecting to external MinGW executable...

Connection established.

------------------------------------------------------------

Package xAct`xTensor`  version 1.1.1, {2014, 9, 28}

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  are welcome to redistribute it under certain conditions. See the General Public License for details.

------------------------------------------------------------
```

Then you define your manifold.

```
DefManifold[M4, 4, {α, β, γ, μ, ν, λ, σ, η}]
```

```
** DefManifold: Defining manifold M4.

** DefVBundle: Defining vbundle TangentM4.
```

```
DefMetric[-1, metric[-α, -β], CD, {";", "∇"}, PrintAs → "g"]
```

```
** DefTensor: Defining symmetric metric tensor metric[-α, -β].
** DefTensor: Defining antisymmetric tensor epsilonmetric[-α, -β, -γ, -η].
** DefTensor: Defining tetrametric Tetrametric[-α, -β, -γ, -η].
** DefTensor: Defining tetrametric Tetrametric†[-α, -β, -γ, -η].
** DefCovD: Defining covariant derivative CD[-α].
** DefTensor: Defining vanishing torsion tensor TorsionCD[α, -β, -γ].
** DefTensor: Defining symmetric Christoffel tensor ChristoffelCD[α, -β, -γ].
** DefTensor: Defining Riemann tensor RiemannCD[-α, -β, -γ, -η].
** DefTensor: Defining symmetric Ricci tensor RicciCD[-α, -β].
** DefCovD:  Contractions of Riemann automatically replaced by Ricci.
** DefTensor: Defining Ricci scalar RicciScalarCD[].
** DefCovD:  Contractions of Ricci automatically replaced by RicciScalar.
** DefTensor: Defining symmetric Einstein tensor EinsteinCD[-α, -β].
** DefTensor: Defining Weyl tensor WeylCD[-α, -β, -γ, -η].
** DefTensor: Defining symmetric TFRicci tensor TFRicciCD[-α, -β].
** DefTensor: Defining Kretschmann scalar KretschmannCD[].
** DefCovD:  Computing RiemannToWeylRules for dim 4
** DefCovD:  Computing RicciToTFRicci for dim 4
** DefCovD:  Computing RicciToEinsteinRules for dim 4
** DefTensor: Defining weight +2 density Detmetric[]. Determinant.
```

Establishing Your Chart

```
<< xAct`xCoba`

------------------------------------------------------------

Package xAct`xCoba`  version 0.8.2, {2014, 9, 28}

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------------------------------------------------------------
```

```
$DefInfoQ = False;
$PrePrint = ScreenDollarIndices;
$CVSimplify = Simplify;
```

```
DefChart[cb, M4, {0, 1, 2, 3}, {t[], r[], θ[], ϕ[]}]
```

```
cb /: CIndexForm[0, cb] := "t";
```

```
cb /: CIndexForm[1, cb] := "r";
```

```
cb /: CIndexForm[2, cb] := "θ";
```

```
cb /: CIndexForm[3, cb] := "ϕ";
```

You should then define any scalar functions and constants you will need for your metric.

```
DefConstantSymbol[M]
```

```
DefConstantSymbol[G]
```

Two Ways to Define Your Metric

```
MatrixForm[met = DiagonalMatrix[{1 - 2 M/r[], (1 - 2 M/r[])^-1, -r[]^2, 2 r[] Sin[θ[]]^2}]]
```

$$\begin{pmatrix} 1-\frac{2M}{r} & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{2M}{r}} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & 2\, r\, \mathrm{Sin}[\theta]^2 \end{pmatrix}$$

MetricInBasis[metric, -cb, met] // TableForm

Added independent rule $g_{tt} \to 1 - \frac{2M}{r}$ for tensor metric

Added independent rule $g_{tr} \to 0$ for tensor metric

Added independent rule $g_{t\theta} \to 0$ for tensor metric

Added independent rule $g_{t\phi} \to 0$ for tensor metric

Added dependent rule $g_{rt} \to g_{tr}$ for tensor metric

Added independent rule $g_{rr} \to \frac{1}{1 - \frac{2M}{r}}$ for tensor metric

Added independent rule $g_{r\theta} \to 0$ for tensor metric

Added independent rule $g_{r\phi} \to 0$ for tensor metric

Added dependent rule $g_{\theta t} \to g_{t\theta}$ for tensor metric

Added dependent rule $g_{\theta r} \to g_{r\theta}$ for tensor metric

Added independent rule $g_{\theta\theta} \to -r^2$ for tensor metric

Added independent rule $g_{\theta\phi} \to 0$ for tensor metric

Added dependent rule $g_{\phi t} \to g_{t\phi}$ for tensor metric

Added dependent rule $g_{\phi r} \to g_{r\phi}$ for tensor metric

Added dependent rule $g_{\phi\theta} \to g_{\theta\phi}$ for tensor metric

Added independent rule $g_{\phi\phi} \to 2\, r\, \text{Sin}[\theta]^2$ for tensor metric

$g_{tt} \to 1 - \frac{2M}{r}$	$g_{tr} \to 0$	$g_{t\theta} \to 0$	$g_{t\phi} \to 0$
$g_{rt} \to 0$	$g_{rr} \to \frac{1}{1-\frac{2M}{r}}$	$g_{r\theta} \to 0$	$g_{r\phi} \to 0$
$g_{\theta t} \to 0$	$g_{\theta r} \to 0$	$g_{\theta\theta} \to -r^2$	$g_{\theta\phi} \to 0$
$g_{\phi t} \to 0$	$g_{\phi r} \to 0$	$g_{\phi\theta} \to 0$	$g_{\phi\phi} \to 2\, r\, \text{Sin}[\theta]^2$

```
TensorValues@metric
```

FoldedRule[

$\{g_{rt} \to g_{tr},\ g_{\theta t} \to g_{t\theta},\ g_{\theta r} \to g_{r\theta},\ g_{\phi t} \to g_{t\phi},\ g_{\phi r} \to g_{r\phi},\ g_{\phi\theta} \to g_{\theta\phi}\}$,

$\{g_{tt} \to 1 - \frac{2M}{r},\ g_{tr} \to 0,\ g_{t\theta} \to 0,\ g_{t\phi} \to 0,\ g_{rr} \to \frac{1}{1 - \frac{2M}{r}},$

$g_{r\theta} \to 0,\ g_{r\phi} \to 0,\ g_{\theta\theta} \to -r^2,\ g_{\theta\phi} \to 0,\ g_{\phi\phi} \to 2\, r\, \mathrm{Sin}[\theta]^2\}$]

```
MetricCompute[metric, cb, "Weyl"[-1, -1, -1, -1]]
```

Now we can explore the second method of defining the metric.

```
g = CTensor[met, {-cb, -cb}];
```

```
SetCMetric[g, -cb];
```

Here was can specify the g_{tt} component,

```
g[{0, -cb}, {0, -cb}]
```

$$1-\frac{2M}{r}$$

```
MetricCompute[g, cb, "Weyl"[-1, -1, -1, -1]];
```

Here we define the covariant derivative,

```
cd = CovDOfMetric[g]
```

$$\text{CCovD}\Big[\text{PDcb}, \text{CTensor}\Big[\Big\{\Big\{\Big\{0, -\frac{M}{2\,M\,r-r^2}, 0, 0\Big\}, \Big\{-\frac{M}{2\,M\,r-r^2}, 0, 0, 0\Big\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\Big\}, \Big\{\Big\{\frac{M\,(2\,M-r)}{r^3}, 0, 0, 0\Big\}, \Big\{0, \frac{M}{2\,M\,r-r^2}, 0, 0\Big\}, \{0, 0, -2\,M+r, 0\}, \Big\{0, 0, 0, \frac{(2\,M-r)\,\text{Sin}[\theta]^2}{r}\Big\}\Big\}, \Big\{\{0, 0, 0, 0\}, \Big\{0, 0, \frac{1}{r}, 0\Big\}, \Big\{0, \frac{1}{r}, 0, 0\Big\}, \Big\{0, 0, 0, \frac{\text{Sin}[2\,\theta]}{r}\Big\}\Big\}, \Big\{\{0, 0, 0, 0\}, \Big\{0, 0, 0, \frac{1}{2\,r}\Big\}, \{0, 0, 0, \text{Cot}[\theta]\}, \Big\{0, \frac{1}{2\,r}, \text{Cot}[\theta], 0\Big\}\Big\}\Big\}, \{\text{cb}, -\text{cb}, -\text{cb}\}, 0\Big], \text{CTensor}\Big[\Big\{\Big\{1-\frac{2\,M}{r}, 0, 0, 0\Big\}, \Big\{0, \frac{1}{1-\frac{2\,M}{r}}, 0, 0\Big\}, \Big\{0, 0, -r^2, 0\Big\}, \Big\{0, 0, 0, 2\,r\,\text{Sin}[\theta]^2\Big\}\Big\}, \{-\text{cb}, -\text{cb}\}, 0\Big]\Big]$$

Christoffel Symbols in a Coordinate Basis

In general we can write the Christoffel symbols

```
Christoffel[CD, PDcb][α, -β, -γ]
```

$\Gamma[\nabla,\mathcal{D}]^{\alpha}{}_{\beta\gamma}$

We can make a table of these in our coordinate basis.

```
Part[TensorValues@ChristoffelCDPDcb, 2] // TableForm
```

$\Gamma[\nabla,\mathcal{D}]^{t}{}_{tt} \to 0$

$\Gamma[\nabla,\mathcal{D}]^{t}{}_{tr} \to -\frac{M}{2Mr-r^2}$

$\Gamma[\nabla,\mathcal{D}]^{t}{}_{t\theta} \to 0$

$\Gamma[\nabla,\mathcal{D}]^{t}{}_{t\phi} \to 0$

$\Gamma[\nabla,\mathcal{D}]^{t}{}_{rr} \to 0$

$\Gamma[\nabla,\mathcal{D}]^{t}{}_{r\theta} \to 0$

$\Gamma[\nabla,\mathcal{D}]^{t}{}_{r\phi} \to 0$

$\Gamma[\nabla,\mathcal{D}]^{t}{}_{\theta\theta} \to 0$

$\Gamma[\nabla,\mathcal{D}]^{t}{}_{\theta\phi} \to 0$

$\Gamma[\nabla,\mathcal{D}]^{t}{}_{\phi\phi} \to 0$

$\Gamma[\nabla,\mathcal{D}]^{r}{}_{tt} \to \frac{M(2M-r)}{r^3}$

$\Gamma[\nabla,\mathcal{D}]^{r}{}_{tr} \to 0$

$\Gamma[\nabla,\mathcal{D}]^{r}{}_{t\theta} \to 0$

$\Gamma[\nabla,\mathcal{D}]^{r}{}_{t\phi} \to 0$

$\Gamma[\nabla,\mathcal{D}]^{r}{}_{rr} \to \frac{M}{2Mr-r^2}$

$\Gamma[\nabla,\mathcal{D}]^{r}{}_{r\theta} \to 0$

$\Gamma[\nabla,\mathcal{D}]^{r}{}_{r\phi} \to 0$

$\Gamma[\nabla,\mathcal{D}]^{r}{}_{\theta\theta} \to -2M+r$

$\Gamma[\nabla,\mathcal{D}]^{r}{}_{\theta\phi} \to 0$

$\Gamma[\nabla,\mathcal{D}]^{r}{}_{\phi\phi} \to \frac{(2M-r)\,\mathrm{Sin}[\theta]^2}{r}$

$\Gamma[\nabla,\mathcal{D}]^{\theta}{}_{tt} \to 0$

$\Gamma[\nabla,\mathcal{D}]^{\theta}{}_{tr} \to 0$

$\Gamma[\nabla,\mathcal{D}]^{\theta}{}_{t\theta} \to 0$

$\Gamma[\nabla,\mathcal{D}]^{\theta}{}_{t\phi} \to 0$

$\Gamma[\nabla,\mathcal{D}]^{\theta}{}_{r r} \to 0$

$\Gamma[\nabla,\mathcal{D}]^{\theta}{}_{r \theta} \to \frac{1}{r}$

$\Gamma[\nabla,\mathcal{D}]^{\theta}{}_{r \phi} \to 0$

$\Gamma[\nabla,\mathcal{D}]^{\theta}{}_{\theta \theta} \to 0$

$\Gamma[\nabla,\mathcal{D}]^{\theta}{}_{\theta \phi} \to 0$

$\Gamma[\nabla,\mathcal{D}]^{\theta}{}_{\phi \phi} \to \frac{\text{Sin}[2\,\theta]}{r}$

$\Gamma[\nabla,\mathcal{D}]^{\phi}{}_{t t} \to 0$

$\Gamma[\nabla,\mathcal{D}]^{\phi}{}_{t r} \to 0$

$\Gamma[\nabla,\mathcal{D}]^{\phi}{}_{t \theta} \to 0$

$\Gamma[\nabla,\mathcal{D}]^{\phi}{}_{t \phi} \to 0$

$\Gamma[\nabla,\mathcal{D}]^{\phi}{}_{r r} \to 0$

$\Gamma[\nabla,\mathcal{D}]^{\phi}{}_{r \theta} \to 0$

$\Gamma[\nabla,\mathcal{D}]^{\phi}{}_{r \phi} \to \frac{1}{2\,r}$

$\Gamma[\nabla,\mathcal{D}]^{\phi}{}_{\theta \theta} \to 0$

$\Gamma[\nabla,\mathcal{D}]^{\phi}{}_{\theta \phi} \to \text{Cot}[\theta]$

$\Gamma[\nabla,\mathcal{D}]^{\phi}{}_{\phi \phi} \to 0$

The Riemann Tensor

riemann = Riemann[cd]

$$\text{CTensor}\Big[\Big\{\Big\{\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},$$

$$\Big\{\Big\{0,\frac{2\,M\,(-2\,M+r)}{r^4},0,0\Big\},\Big\{\frac{2\,M}{(2\,M-r)\,r^2},0,0,0\Big\},\{0,0,0,0\},\{0,0,0,0\}\Big\},$$

$$\Big\{\Big\{0,0,\frac{M\,(2\,M-r)}{r^4},0\Big\},\{0,0,0,0\},\Big\{-\frac{M}{r},0,0,0\Big\},\{0,0,0,0\}\Big\},$$

$$\Big\{\Big\{0,0,0,\frac{M\,(2\,M-r)}{2\,r^4}\Big\},\{0,0,0,0\},\{0,0,0,0\},\Big\{\frac{M\,\text{Sin}[\theta]^2}{r^2},0,0,0\Big\}\Big\}\Big\},$$

$$\Big\{\Big\{\Big\{0,-\frac{2\,M\,(-2\,M+r)}{r^4},0,0\Big\},\Big\{-\frac{2\,M}{(2\,M-r)\,r^2},0,0,0\Big\},\{0,0,0,0\},\{0,0,0,0\}\Big\},$$

$$\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},$$

$$\Big\{\{0,0,0,0\},\Big\{0,0,\frac{M}{(2\,M-r)\,r^2},0\Big\},\Big\{0,-\frac{M}{r},0,0\Big\},\{0,0,0,0\}\Big\},$$

$$\Big\{\{0,0,0,0\},\Big\{0,0,0,\frac{-4\,M+r}{4\,r^2\,(-2\,M+r)}\Big\},\Big\{0,0,0,\frac{\text{Cot}[\theta]}{2\,r}\Big\},$$

$$\Big\{0,\frac{(4\,M-r)\,\text{Sin}[\theta]^2}{2\,r^2},\frac{\text{Cos}[\theta]\,\text{Sin}[\theta]}{r^2},0\Big\}\Big\}\Big\},$$

$$\Big\{\Big\{\Big\{0,0,-\frac{M\,(2\,M-r)}{r^4},0\Big\},\{0,0,0,0\},\Big\{\frac{M}{r},0,0,0\Big\},\{0,0,0,0\}\Big\},$$

$$\Big\{\{0,0,0,0\},\Big\{0,0,-\frac{M}{(2\,M-r)\,r^2},0\Big\},\Big\{0,\frac{M}{r},0,0\Big\},\{0,0,0,0\}\Big\},$$

$$\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},$$

$$\Big\{\{0,0,0,0\},\Big\{0,0,0,\frac{\text{Cot}[\theta]}{2\,r}\Big\},\Big\{0,0,0,\frac{3}{2}-\frac{M}{r}\Big\},$$

$$\Big\{0,\frac{\text{Cos}[\theta]\,(2\,M-r)\,\text{Sin}[\theta]}{r},\frac{(-2\,M+3\,r)\,\text{Sin}[\theta]^2}{r^2},0\Big\}\Big\}\Big\},$$

$$\Big\{\Big\{\Big\{0,0,0,-\frac{M\,(2\,M-r)}{2\,r^4}\Big\},\{0,0,0,0\},\{0,0,0,0\},\Big\{-\frac{M\,\text{Sin}[\theta]^2}{r^2},0,0,0\Big\}\Big\},$$

$$\Big\{\{0,0,0,0\},\Big\{0,0,0,-\frac{-4\,M+r}{4\,r^2\,(-2\,M+r)}\Big\},\Big\{0,0,0,-\frac{\text{Cot}[\theta]}{2\,r}\Big\},$$

$$\Big\{0,-\frac{(4\,M-r)\,\text{Sin}[\theta]^2}{2\,r^2},-\frac{\text{Cos}[\theta]\,\text{Sin}[\theta]}{r^2},0\Big\}\Big\},\Big\{\{0,0,0,0\},\Big\{0,0,0,-\frac{\text{Cot}[\theta]}{2\,r}\Big\},$$

$$\Big\{0,0,0,-\frac{3}{2}+\frac{M}{r}\Big\},\Big\{0,-\frac{\text{Cos}[\theta]\,(2\,M-r)\,\text{Sin}[\theta]}{r},-\frac{(-2\,M+3\,r)\,\text{Sin}[\theta]^2}{r^2},0\Big\}\Big\},$$

$$\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\}\Big\}\Big\},\{-cb,-cb,-cb,cb\},0\Big]$$

riemann[{0, -cb}, {1, -cb}, {0, -cb}, {1, cb}]

$$\frac{2\,M\,(-2\,M+r)}{r^4}$$

riemann[{3, -cb}, {2, -cb}, {3, -cb}, {2, cb}]

$$-\frac{(-2\,M+3\,r)\,\text{Sin}[\theta]^2}{r^2}$$

The Ricci Tensor and Ricci Scalar

```
Ricci[cd][-α, -β]
```

$$\begin{pmatrix} \frac{M(-2M+r)}{2r^4} & 0 & 0 & 0 \\ 0 & -\frac{1}{8Mr-4r^2} & \frac{\text{Cot}[\theta]}{2r} & 0 \\ 0 & \frac{\text{Cot}[\theta]}{2r} & \frac{3}{2}+\frac{M}{r} & 0 \\ 0 & 0 & 0 & -\frac{(2M+5r)\,\text{Sin}[\theta]^2}{2r^2} \end{pmatrix}_{\alpha\beta}$$

```
Ricci[cd][{0, -cb}, {0, -cb}]
```

$$\frac{M(-2M+r)}{2r^4}$$

```
rs = RicciScalar[cd][[1]]
```

$$-\frac{2M+5r}{2r^3}$$

The Einstein Tensor

```
Einstein[cd][-α, -β]
```

$$\begin{pmatrix} -\frac{(2 M-r)\ (4 M+5 r)}{4 r^4} & 0 & 0 & 0 \\ 0 & -\frac{M+3 r}{4 M r^2-2 r^3} & \frac{\text{Cot}[\theta]}{2 r} & 0 \\ 0 & \frac{\text{Cot}[\theta]}{2 r} & \frac{2 M+r}{4 r} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{\alpha\beta}$$

```
Einstein[cd][{0, -cb}, {0, -cb}]
```

$$-\frac{(2 M-r)\ (4 M+5 r)}{4 r^4}$$

The Stress Energy Tensor

We next need to calculate the stress-energy tensor. we begin by defining the density field.

```
DefTensor[ρ, M4]
```

```
DefTensor[ρ, M4, GenSet[]]
```

Here we define the 4-velocity.

```
U = CTensor[{1, 0, 0, 0}, {-cb}]
```

```
CTensor[{1, 0, 0, 0}, {-cb}, 0]
```

Here is the stress-energy tensor for a pressure-less dust

```
Td[α_, β_] := ρ[] U[-α] U[-β]
```

```
Td[α, β]
```

$$\begin{pmatrix} \rho[\,] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \beta\alpha$$

Here we have the T_{tt} component.

```
Td[{0, cb}, {0, cb}]
```

$\rho[\,]$

The pressure-less dust is very simple. A little more complicated is the perfect fluid. This requires us to define a pressure field.

DefTensor[p, M4]

DefTensor[p, M4, GenSet[]]

The stress-energy tensor for this situation is,

Tf[α_, β_] := (ρ[] + p[]) U[-α] U[-β] + p[] g[-α, -β]

Tf[α, β]

$$\begin{pmatrix} p[]+p[]\left(1-\frac{2M}{r}\right)+\rho[] & 0 & 0 & 0 \\ 0 & \frac{p[]}{1-\frac{2M}{r}} & 0 & 0 \\ 0 & 0 & -p[]\ r^2 & 0 \\ 0 & 0 & 0 & 2\ p[]\ r\ \mathrm{Sin}[\theta]^2 \end{pmatrix}^{\alpha\beta}$$

Tf[{-0, cb}, {-0, cb}]

$$p[] + p[]\left(1-\frac{2M}{r}\right) + \rho[]$$

Tf[{-1, cb}, {-1, cb}]

$$\frac{p[]}{1-\frac{2M}{r}}$$

Einstein's Field Equations

We will now try to write Einstein's equation for the tt components of the Einstein and stress-energy tensors. We begin with this formulation,

$$R_{tt} - \frac{1}{2} R\, g_{tt} = 8 \pi G\, T_{tt}$$

```
tteq = Ricci[cd][{0, -cb}, {0, -cb}] - 1/2 rs g[{0, -cb}, {0, -cb}] ==
  8 π G Tf[{-0, cb}, {-0, cb}]
```

$$\frac{M\,(-2\,M+r)}{2\,r^4}+\frac{\left(1-\frac{2\,M}{r}\right)\,(2\,M+5\,r)}{4\,r^3} == 8\,G\,\pi\,\left(p[\,]+p[\,]\,\left(1-\frac{2\,M}{r}\right)+\rho[\,]\right)$$

```
tteq // FullSimplify
```

$$-\frac{(2\,M-r)\,(4\,M+5\,r)}{4\,r^4} == 8\,G\,\pi\,\left(p[\,]\,\left(2-\frac{2\,M}{r}\right)+\rho[\,]\right)$$

We can also write the equation,

$$G_{tt} = 8\,\pi\,G\,T_{tt}$$

```
tteq2 = Einstein[cd][{0, -cb}, {0, -cb}] == 8 π G Tf[{-0, cb}, {-0, cb}] // FullSimplify
```

$$-\frac{(2\,M-r)\,(4\,M+5\,r)}{4\,r^4} == 8\,G\,\pi\,\left(p[\,]\,\left(2-\frac{2\,M}{r}\right)+\rho[\,]\right)$$

We can genralize this

```
eineq[a_, b_] :=
 Einstein[cd][{a, -cb}, {b, -cb}] - 8 π G Tf[{-a, cb}, {-b, cb}] // FullSimplify
```

```
eineq[0, 0]
```

$$-\frac{(2M-r)(4M+5r)}{4r^4}-8G\pi\left(p[\,]\left(2-\frac{2M}{r}\right)+\rho[\,]\right)$$

```
eineq[1, 1]
```

$$\frac{M+3r-16G\pi p[\,]r^3}{2r^2(-2M+r)}$$

```
eineq[0, 1]
```

0

```
Table[eineq[a, b], {a, 0, 3}, {b, 0, 3}] // TableForm
```

$-\frac{(2M-r)(4M+5r)}{4r^4} - 8G\pi\left(p[\,]\left(2-\frac{2M}{r}\right)+\rho[\,]\right)$	0	0	0
0	$\frac{M+3r-16G\pi p[\,]r^3}{2r^2(-2M+r)}$	$\frac{\text{Cot}[\theta]}{2r}$	0
0	$\frac{\text{Cot}[\theta]}{2r}$	$\frac{1}{4}+\frac{M}{2r}+8G\pi p[\,]r^2$	0
0	0	0	$-16G\pi p[\,]r\,\text{Sin}[\theta]^2$

Thank You!